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Two-Loop QCD Corrections to the Helicity Amplitudes for $H \to 3$ partons

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Keywords: QCD, Higgs, NLO and NNLO calculations.

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1. Introduction

Within the Standard Model of particle physics, the Higgs boson is the only particle remaining to be discovered. The Higgs boson is crucial for electroweak symmetry breaking, the mechanism that explains the generation of the masses of the fermions and the weak gauge bosons. While the vacuum expectation value of the Higgs field is directly related to the Fermi constant, its mass remains a free parameter that can be constrained but not predicted by the theory.

The direct detection of the Higgs boson at LEP and the Tevatron has been a very challenging task over the past two decades [1,2]. The LEP experiments [1] excluded Higgs boson masses below $M_H \sim 114$ GeV, while the Tevatron excluded Higgs masses in a narrow window around the W-pair threshold $M_H \sim 2M_W$. With the start of the Large Hadron Collider (LHC) at CERN, the focus of the Higgs boson searches has moved to the ATLAS and CMS experiments, where the search is based on different decay channels. For $M_H \gtrsim 135$ GeV the decay into two weak gauge bosons is most prominent, while for lower

Higgs boson masses the search at the LHC is much more challenging, since the dominant decay modes are overwhelmed by large Standard Model backgrounds. For example, the search for light Higgs bosons with $M_H \lesssim 130$ GeV is based on the rare decay $H \to \gamma\gamma$, which has a branching ratio of $\mathcal{O}\left(10^{-3}\right)$, thus requiring larger integrated luminosity. For light Higgs boson masses, the dominant production process at the LHC is gluon fusion. Based on the first $\mathcal{O}(5\ \mathrm{fb}^{-1})$ of proton data taken in 2011, ATLAS [3] and CMS [4] are now able to narrow down the allowed mass range for the Standard Model Higgs boson considerably by essentially excluding the Higgs bosons in the range $\mathcal{O}(130\ \mathrm{GeV}) \gtrsim M_H \gtrsim \mathcal{O}(600\ \mathrm{GeV})$, while observing an excess of Higgs boson candidate events around $M_H = 125$ GeV.

At leading-order (LO), the Higgs coupling to the two gluons is mediated through a quark loop. Since the Higgs coupling to the quarks is proportional to the quark masses, the dominant contribution is generated from the top quark [5]. The next-to-leading-order (NLO) corrections [6] to this process have been calculated and turn out to be very large (>60%). In the heavy top quark limit, $M_t \to \infty$, the Hgg coupling becomes independent of M_t. One can therefore integrate out the top mass (M_t) and formulate an effective Langrangian $\mathcal{L}_{eff}$ for the Hgg coupling [7]. This technique is valid for $M_H < 2M_t$ and reduces the loops that need to be calculated by one. In this limit, the inclusive Higgs boson production cross section has been computed at NLO [8] and at next-to-next-to-leading-order (NNLO) [9], indicating a stabilization of the perturbative prediction at this order.

Experimental searches of the Higgs boson apply final state cuts to improve the significance of a potential signal over Standard Model background processes. To implement these cuts in the theoretical description, fully exclusive calculations, which keep track of the kinematical information of all final state particles (Higgs decay products and QCD radiation) are mandatory. In the heavy top quark limit, the NNLO corrections to Higgs production via gluon fusion have been computed fully exclusively, including the Higgs decay to two photons or two weak gauge bosons by two independent groups [10, 11]. These calculations are in the form of flexible parton-level event generators, which can properly account for the final state restrictions used in the experimental studies.

An important final state discriminator is the number of jets observed in addition to the potential Higgs boson decay products, and the Higgs signal can often be enhanced by applying jet vetos [12, 13]. In many searches, it is however expected that the $H + 0j$ and $H + 1j$ samples contribute roughly equally to the sensitivity. In the above-mentioned NNLO calculations, the $H + 1j$ final states are included to NLO [14], and the $H + 2j$ final states to LO. NLO corrections to $H + 2j$-production have been derived recently [15]. The correlation between samples of different jet multiplicity has recently been matter of quite some debate [16,17], and an improved theoretical description of $H+1j$-production to NNLO accuracy is essential in order to have the same theoretical accuracy for the $H + 1j$ contribution as for the $H + 0j$ contribution.

In the heavy top quark limit, a full NNLO QCD calculation of $H + 1j$ production requires the computation of the matrix elements of three contributions:

(a) the tree level $H \to 5$ partons amplitudes,

(b) the one-loop corrections to the $H \to 4$ partons amplitudes,

(c) the two-loop corrections to the $H \to ggg$ and $H \to q\bar{q}g$ matrix elements.

The tree-level contributions of type (a) can be computed with standard tree-level methods, and compact expressions can be obtained by using MHV-techniques [18]. The one-loop terms of type (b) were derived in an analytic form in [19], and form part of the NLO corrections to $H + 2j$ final states. The $H \to ggg$ and $H \to gq\bar{q}$ matrix elements were previously known to one loop [20]. In this paper, we compute the two-loop corrections to these processes in the heavy top quark limit. We note that expressions for the two-loop $H \to ggg$ helicity amplitudes were previously given in the PhD thesis of one of us. [21]

The three different contributions must be combined into a parton-level event generator program. All three are separately infrared-divergent, and only their sum is finite and physically meaningful. To combine the contributions, an infrared subtraction method is required. Several methods have been applied successfully in NNLO calculations of exclusive observables in the recent past: sector decomposition [22], q_T-subtraction [23] and antenna subtraction [24–26]. A resulting parton-level event generator will allow an NNLO description of both $H + 1j$ production and of the Higgs boson transverse momentum distribution.

In addition to their phenomenological importance for precise predictions of collider processes, two-loop amplitudes are interesting to investigate fundamental properties of quantum field theory at high perturbative orders, aiming to identify regularities, asymptotic behaviour and heading towards an all-order understanding of field theory amplitudes. In this context, the massless $2 \to 2$ QCD scattering amplitudes at two loops [27] were used to determine the high energy Regge trajectories of quarks and gluons at the two-loop level [28]. Similarly, the two-loop decay matrix elements for $\gamma^* \to q\bar{q}g$ [29] and $H \to 3$ partons fix the the two-loop splitting amplitudes [30], which describe the collinear factorization of loop amplitudes.

This paper is structured as follows: in Section 2, we consider the effective coupling of the Higgs boson to light partons and define the kinematics. In Section 3, we describe the method used to construct the tensor coefficients in the general amplitudes. In Section 4, we construct the helicity amplitudes. The derivation of the two-loop corrections to the helicity amplitudes, their renormalization and infrared properties are described in Section 5. We conclude with an outlook in Section 6. The one-loop and two-loop helicity amplitudes can be expressed in compact analytic form, and are enclosed in the Appendix.

2. Notation and kinematics

2.1 The effective Lagrangian

At tree level, the Higgs boson does not couple either to the gluon or to massless quarks. In higher orders in perturbation theory, heavy quark loops introduce a coupling between the Higgs boson and gluons. As we mentioned in Sec. 1, in the heavy top quark limit, $M_t \to \infty$, the Hgg coupling becomes independent of M_t. We can therefore integrate out the top quark field and formulate an effective Lagrangian, $\mathcal{L}_{eff}$ [7] that couples the scalar Higgs field and the gluon field strength tensor, thereby approximating the Hgg coupling. This large top quark mass approximation has been shown to work very well under the condition that the kinematic scales involved are smaller than twice the top quark mass [31].

The effective Lagrangian reads,

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4} H G_a^{\mu\nu} G_{a,\mu\nu} \ . \tag{2.1}$$

where $G_a^{\mu\nu}$ is the field strength tensor of the gluon. The coupling λ has inverse mass dimension. It can be computed by matching [32,33] the effective theory to the full standard model cross section [6].

2.2 Kinematics

We consider the decay of the Higgs boson to three gluons,

$$H(p_4) \longrightarrow g_1(p_1) + g_2(p_2) + g_3(p_3) \ , \tag{2.1}$$

or into a quark-antiquark pair and a gluon,

$$H(p_4) \longrightarrow q(p_1) + \bar{q}(p_2) + g_(p_3) \ . \tag{2.2}$$

It is convenient to define the invariants,

$$s_{12} = (p_1 + p_2)^2 \ , \qquad s_{13} = (p_1 + p_3)^2 \ , \qquad s_{23} = (p_2 + p_3)^2 \ , \tag{2.3}$$

which fulfill

$$p_4^2 = s_{12} + s_{13} + s_{23} \equiv s_{123} \equiv M_H^2 \ , \tag{2.4}$$

as well as the dimensionless invariants,

$$x = s_{12}/s_{123} \ , \qquad y = s_{13}/s_{123} \ , \qquad z = s_{23}/s_{123} \ , \tag{2.5}$$

which satisfy $x + y + z = 1$.

3. The general tensors

The amplitudes $|\mathcal{M}\rangle$ can be written as,

$$\begin{aligned} |\mathcal{M}_{ggg}\rangle &= S_{\mu\nu\rho}(g_1; g_2; g_3)\epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \, , \\ |\mathcal{M}_{q\bar{q}g}\rangle &= T_\rho(q, \bar{q}; g)\epsilon^\rho \, , \end{aligned} \tag{3.1}$$

while the partonic currents may be perturbatively decomposed as,

$$\begin{aligned} S_{\mu\nu\rho}(g_1; g_2; g_3) &= \lambda\sqrt{4\pi\alpha_s} f^{a_1 a_2 a_3} \Big[S^{(0)}_{\mu\nu\rho}(g_1; g_2; g_3) + \left(\frac{\alpha_s}{2\pi}\right) S^{(1)}_{\mu\nu\rho}(g_1; g_2; g_3) \\ &\quad + \left(\frac{\alpha_s}{2\pi}\right)^2 S^{(2)}_{\mu\nu\rho}(g_1; g_2; g_3) + \mathcal{O}(\alpha_s^3) \Big], \end{aligned} \tag{3.2}$$

$$\begin{aligned} T_\rho(q; \bar{q}; g) &= \lambda\sqrt{4\pi\alpha_s} T^a_{ij} \Big[T^{(0)}_\rho(q; \bar{q}; g) + \left(\frac{\alpha_s}{2\pi}\right) T^{(1)}_\rho(q; \bar{q}; g) \\ &\quad + \left(\frac{\alpha_s}{2\pi}\right)^2 T^{(2)}_\rho(q; \bar{q}; g) + \mathcal{O}(\alpha_s^3) \Big], \end{aligned} \tag{3.3}$$

where α_s is the QCD coupling constant, and $S^{(i)}_{\mu\nu\rho}$ and $T^{(i)}_\rho$ are the i-loop contributions to the amplitude. The SU(3) generators are normalized as $tr(T^a T^b) = \delta^{ab}/2$.

3.1 The general tensor for $H \to ggg$

The most general tensor structure for the partonic current $\mathcal{S}_{\mu\nu\rho}(g_1; g_2; g_3)$ is given by,

$$
\begin{aligned}
S_{\mu\nu\rho}(g_1;g_2;g_3)\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\rho = & \sum_{i,j,k=1}^{3} A_{i\,j\,k}\, p_i\cdot\epsilon_1\, p_j\cdot\epsilon_2\, p_k\cdot\epsilon_3 + \sum_{i=1}^{3} B_i\, p_i\cdot\epsilon_1\, \epsilon_2\cdot\epsilon_3 \\
& + \sum_{i=1}^{3} C_i\, p_i\cdot\epsilon_2\, \epsilon_1\cdot\epsilon_3 + \sum_{i=1}^{3} D_i\, p_i\cdot\epsilon_3\, \epsilon_1\cdot\epsilon_2 \\
& = A_{211}\, p_2\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_1\cdot\epsilon_3 + A_{212}\, p_2\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_2\cdot\epsilon_3 + A_{231}\, p_2\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_1\cdot\epsilon_3 \\
& + A_{232}\, p_2\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_2\cdot\epsilon_3 + A_{311}\, p_3\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_1\cdot\epsilon_3 + A_{312}\, p_3\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_2\cdot\epsilon_3 \\
& + A_{331}\, p_3\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_1\cdot\epsilon_3 + A_{332}\, p_3\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_2\cdot\epsilon_3 \\
& + B_2\, \epsilon_2\cdot\epsilon_3\, p_2\cdot\epsilon_1 + B_3\, \epsilon_2\cdot\epsilon_3\, p_3\cdot\epsilon_1 \\
& + C_1\, \epsilon_1\cdot\epsilon_3\, p_1\cdot\epsilon_2 + C_3\, \epsilon_1\cdot\epsilon_3\, p_3\cdot\epsilon_2 \\
& + D_1\, \epsilon_1\cdot\epsilon_2\, p_1\cdot\epsilon_3 + D_2\, \epsilon_1\cdot\epsilon_2\, p_2\cdot\epsilon_3\,,
\end{aligned} \tag{3.4}
$$

where the constraints $p_1 \cdot \epsilon_1 = 0$, $p_2 \cdot \epsilon_2 = 0$ and $p_3 \cdot \epsilon_3 = 0$ have been applied. The tensor must satisfy the QCD Ward identity when the gluon polarization vectors ϵ_1, ϵ_2 and ϵ_3 are replaced with the respective gluon momentum,

$$
\begin{aligned}
(\epsilon_1 \to p_1) &\to S_{\mu\nu\rho}(g_1;g_2;g_3)p_1^\mu\epsilon_2^\nu\epsilon_3^\rho = 0\,, \\
(\epsilon_2 \to p_2) &\to S_{\mu\nu\rho}(g_1;g_2;g_3)\epsilon_1^\mu p_2^\nu\epsilon_3^\rho = 0\,, \\
(\epsilon_3 \to p_3) &\to S_{\mu\nu\rho}(g_1;g_2;g_3)\epsilon_1^\mu\epsilon_2^\nu p_3^\rho = 0\,.
\end{aligned} \tag{3.5}
$$

These constraints yield relations amongst the 14 distinct tensor structures and applying these identities give the gauge invariant form of the tensor,

$$
S_{\mu\nu\rho}(g_1;g_2;g_3)\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\rho = A_{211}T_{211} + A_{311}T_{311} + A_{232}T_{232} + A_{312}T_{312}\,, \tag{3.6}
$$

where A_{ijk} are gauge independent functions and the tensor structures T_{ijk} are given by,

$$
\begin{aligned}
T_{232} &= p_2\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_2\cdot\epsilon_3 - \frac{1}{2}\,\epsilon_2\cdot\epsilon_3\, p_2\cdot\epsilon_1\, s_{23} - \frac{p_3\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_2\cdot\epsilon_3\, s_{12}}{s_{13}} + \frac{1}{2}\,\frac{\epsilon_2\cdot\epsilon_3\, p_3\cdot\epsilon_1\, s_{23}\, s_{12}}{s_{13}}, \\
T_{211} &= p_2\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_1\cdot\epsilon_3 - \frac{1}{2}\,\epsilon_1\cdot\epsilon_2\, p_1\cdot\epsilon_3\, s_{12} - \frac{p_2\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_2\cdot\epsilon_3\, s_{13}}{s_{23}} + \frac{1}{2}\,\frac{\epsilon_1\cdot\epsilon_2\, p_2\cdot\epsilon_3\, s_{13}\, s_{12}}{s_{23}}, \\
T_{311} &= p_3\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_1\cdot\epsilon_3 - \frac{1}{2}\,\epsilon_1\cdot\epsilon_3\, p_1\cdot\epsilon_2\, s_{13} - \frac{p_3\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_1\cdot\epsilon_3\, s_{12}}{s_{23}} + \frac{1}{2}\,\frac{\epsilon_1\cdot\epsilon_3\, p_3\cdot\epsilon_2\, s_{13}\, s_{12}}{s_{23}}, \\
T_{312} &= p_3\cdot\epsilon_1\, p_1\cdot\epsilon_2\, p_2\cdot\epsilon_3 - p_2\cdot\epsilon_1\, p_3\cdot\epsilon_2\, p_1\cdot\epsilon_3 + \frac{1}{2}\,\epsilon_1\cdot\epsilon_3\, p_3\cdot\epsilon_2\, s_{12} + \frac{1}{2}\,\epsilon_1\cdot\epsilon_2\, p_1\cdot\epsilon_3\, s_{23} \\
&\quad - \frac{1}{2}\,\epsilon_1\cdot\epsilon_3\, p_1\cdot\epsilon_2\, s_{23} + \frac{1}{2}\,\epsilon_2\cdot\epsilon_3\, p_2\cdot\epsilon_1\, s_{13} - \frac{1}{2}\,\epsilon_1\cdot\epsilon_2\, p_2\cdot\epsilon_3\, s_{13} - \frac{1}{2}\,\epsilon_2\cdot\epsilon_3\, p_3\cdot\epsilon_1\, s_{12}\,.
\end{aligned} \tag{3.7}
$$

The coefficients are functions of the invariants s_{12}, s_{23} and s_{13} and are further related by symmetry under the interchange of the three gluons,

$$
\begin{aligned}
A_{211}(s_{12}, s_{13}, s_{23}) &= -A_{311}(s_{13}, s_{12}, s_{23})\,, \\
A_{232}(s_{12}, s_{13}, s_{23}) &= -A_{311}(s_{12}, s_{23}, s_{13})\,.
\end{aligned} \tag{3.8}
$$

The coefficients A_{ijk} may be easily extracted from a Feynman diagram calculation using projectors such that,

$$\sum_{\rm spins} \mathcal{P}(A_{ijk})\, S_{\mu\nu\rho}(g_1;g_2;g_3)\epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho = A_{ijk}\,, \tag{3.9}$$

where the four projectors are given by,

$$\begin{aligned}
\mathcal{P}(A_{311}) &= -\frac{(D-4)}{s_{12}\,s_{23}\,s_{13}\;(D-3)}T^\dagger_{232} - \frac{s_{23}\;(D-4)}{{s_{13}}^2{s_{12}}^2\,(D-3)}T^\dagger_{211}\\
&\quad+\frac{s_{23}\,D}{s_{12}\,{s_{13}}^3\,(D-3)}T^\dagger_{311} - \frac{(D-2)}{{s_{13}}^2 s_{12}\;(D-3)}T^\dagger_{312},\\
\mathcal{P}(A_{232}) &= \frac{s_{13}\,D}{s_{12}\,{s_{23}}^3\,(D-3)}T^\dagger_{232} + \frac{(D-4)}{s_{23}\,{s_{12}}^2\,(D-3)}T^\dagger_{211}\\
&\quad-\frac{(D-4)}{s_{12}\,s_{23}\,s_{13}\;(D-3)}T^\dagger_{311} + \frac{(D-2)}{{s_{23}}^2 s_{12}\;(D-3)}T^\dagger_{312},\\
\mathcal{P}(A_{312}) &= \frac{(D-2)}{{s_{23}}^2 s_{12}\;(D-3)}T^\dagger_{232} + \frac{(D-2)}{s_{13}\,{s_{12}}^2\,(D-3)}T^\dagger_{211}\\
&\quad-\frac{(D-2)}{{s_{13}}^2 s_{12}\;(D-3)}T^\dagger_{311} + \frac{D}{s_{12}\,s_{23}\,s_{13}\;(D-3)}T^\dagger_{312},\\
\mathcal{P}(A_{211}) &= \frac{(D-4)}{s_{23}\,{s_{12}}^2\,(D-3)}T^\dagger_{232} + \frac{s_{23}\,D}{s_{13}\,{s_{12}}^3\,(D-3)}T^\dagger_{211}\\
&\quad-\frac{s_{23}\;(D-4)}{{s_{13}}^2{s_{12}}^2\,(D-3)}T^\dagger_{311} + \frac{(D-2)}{s_{13}\,{s_{12}}^2\,(D-3)}T^\dagger_{312}.
\end{aligned} \tag{3.10}$$

Each of the tensor coefficients A_{ijk} has a perturbative expansion of the form,

$$A_{ijk} = \lambda\sqrt{4\pi\alpha_s}\left[A^{(0)}_{ijk} + \left(\frac{\alpha_s}{2\pi}\right)A^{(1)}_{ijk} + \left(\frac{\alpha_s}{2\pi}\right)^2 A^{(2)}_{ijk} + \mathcal{O}\left((\alpha_s)^3\right)\right]\,, \tag{3.11}$$

while the tree-level values are,

$$\begin{aligned}
A^{(0)}_{211} &= \frac{2}{s_{13}}\,,\\
A^{(0)}_{311} &= -\frac{2}{s_{12}}\,,\\
A^{(0)}_{232} &= \frac{2}{s_{12}}\,,\\
A^{(0)}_{312} &= -\frac{2}{s_{12}} - \frac{2}{s_{23}} - \frac{2}{s_{13}}\,.
\end{aligned} \tag{3.12}$$

3.2 The general tensor for $H \to q\bar{q}g$

The most general tensor structure for the partonic current $T_\rho(q;\bar{q};g)$ is given by,

$$T_\rho(q;\bar{q};g)\epsilon_3^\rho = A_1\bar{u}(p_1)\not{p}_3 v(p_2)p_1\cdot\epsilon_3 + A_2\bar{u}(p_1)\not{p}_3 v(p_2)p_2\cdot\epsilon_3 + A_3\bar{u}(p_1)\not{\epsilon}_3 v(p_2), \tag{3.13}$$

where $p_3 \cdot \epsilon_3 = 0$ has been applied. The QCD Ward identity yields,

$$A_3 = -p_1 \cdot p_3 A_1 - p_2 \cdot p_3 A_2 \, ,$$

such that the amplitude can be written as,

$$\begin{aligned} T_\rho(q;\bar{q};g)\epsilon_3^\rho = & \; A_1 \left(\bar{u}(p_1) \not{p}_3 v(p_2) p_2 \cdot \epsilon_3 - \bar{u}(p_1) \not{\epsilon}_3 v(p_2) p_2 \cdot p_3 \right) \\ & + A_2 \left(\bar{u}(p_1) \not{p}_3 v(p_2) p_1 \cdot \epsilon_3 - \bar{u}(p_1) \not{\epsilon}_3 v(p_2) p_1 \cdot p_3 \right) \qquad (3.14) \\ \equiv & \; A_1 T_1 + A_2 T_2 \, . \qquad (3.15) \end{aligned}$$

The coefficients A_i can be extracted from a Feynman diagram calculation by using projectors such that,

$$\sum_{\text{spins}} \mathcal{P}(A_i) T_\rho(q;\bar{q};g)\epsilon_3^\rho = A_i \, , \qquad (3.16)$$

where the projectors are given by,

$$\mathcal{P}(A_1) = \frac{(D-2)}{2(D-3)s_{12}s_{13}^2} T_1^\dagger - \frac{(D-4)}{2(D-3)s_{12}s_{13}s_{23}} T_2^\dagger \, , \qquad (3.17)$$

$$\mathcal{P}(A_2) = -\frac{(D-4)}{2(D-3)s_{12}s_{13}s_{23}} T_1^\dagger + \frac{(D-2)}{2(D-3)s_{12}s_{23}^2} T_2^\dagger \, . \qquad (3.18)$$

Each of the coefficients A_i has a perturbative expansion of the form,

$$A_i = \lambda \sqrt{4\pi\alpha_s} \left[A_i^{(0)} + \left(\frac{\alpha_s}{2\pi}\right) A_i^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 A_i^{(2)} + \mathcal{O}\left((\alpha_s)^3\right) \right] \, , \qquad (3.19)$$

while the tree-level values are simply,

$$A_1^{(0)} = A_2^{(0)} = \frac{1}{s_{12}} \, . \qquad (3.20)$$

4. Helicity amplitudes

The general form of the renormalized helicity amplitude $|\mathcal{M}_{ggg}^{\lambda_1\lambda_2\lambda_3}\rangle$ for the decay, $H(p_4) \to g_1(p_1,\lambda_1) + g_2(p_2,\lambda_2) + g_3(p_3,\lambda_3)$ can be written as,

$$|\mathcal{M}_{ggg}^{\lambda_1\lambda_2\lambda_3}\rangle = S_{\mu\nu\rho}(g_1;g_2;g_3)\epsilon_{1,\lambda_1}^\mu(p_1)\epsilon_{2,\lambda_2}^\nu(p_2)\epsilon_{3,\lambda_3}^\rho(p_3) \, , \qquad (4.1)$$

where the $\lambda_i = \pm$ denote helicity. Similarly, the amplitude for the decay $|\mathcal{M}_{q\bar{q}g}^{\lambda_1\lambda_2\lambda_3}\rangle$ for the decay, $H(p_4) \to q(p_1,\lambda_1) + \bar{q}(p_2,\lambda_2) + g(p_3,\lambda_3)$ can be written as,

$$|\mathcal{M}_{q\bar{q}g}^{\lambda_1\lambda_2\lambda_3}\rangle = T_\rho(q^{\lambda_1};\bar{q}^{\lambda_2};g)\epsilon_{3,\lambda_3}^\rho(p_3) \, . \qquad (4.2)$$

The helicity amplitudes can be obtained from the general D-dimensional tensors of Eqs. (3.4) and (3.13) by setting the dimensionality of the Lorentz matrices to be four and using standard four-dimensional helicity techniques [34–36]. This corresponds to working in the 't Hooft-Veltman scheme. We use the standard convention of denoting the two helicity states of a four-dimensional light-like spinor $\psi(p)$ by,

$$\psi_{\pm}(p) = \frac{1}{2}(1 \pm \gamma_5)\psi(p), \tag{4.3}$$

with the further notation,

$$|p\pm\rangle = \psi_{\pm}(p), \qquad \langle p\pm| = \overline{\psi}_{\pm}(p). \tag{4.4}$$

Particles may thus be crossed to the initial state by reversing the sign of the helicity. The basic quantity is the spinor product,

$$\langle pq\rangle = \langle p-|q+\rangle, \qquad [pq] = [p+|q-], \tag{4.5}$$

such that

$$\langle pq\rangle[qp] = 2p\cdot q. \tag{4.6}$$

The polarization vector of a outgoing light-like particle with momentum p can then be written as

$$\epsilon^{\mu}_{\pm}(p;q) = \pm\frac{\langle q\mp|\gamma^{\mu}|p\mp\rangle}{\sqrt{2}\langle q\mp|p\pm\rangle} \tag{4.7}$$

where q is a light-like reference momentum that satisfies $q\cdot p \neq 0$ but which otherwise can be chosen freely.

Important identities relating spinorial objects are the Fierz rearrangement,

$$\langle p+|\gamma^{\mu}|q+\rangle\langle r+|\gamma^{\mu}|s+\rangle = 2[pr]\langle sq\rangle \tag{4.8}$$

and charge conjugation,

$$\langle p+|\gamma^{\mu}|q+\rangle = \langle q-|\gamma^{\mu}|p-\rangle. \tag{4.9}$$

Substituting Eq. (3.7) into Eq. (3.6), we can express the helicity amplitudes for $H\to ggg$ directly in terms of spinor products. It turns out that the only two independent helicity amplitudes are $|\mathcal{M}^{+++}_{ggg}\rangle$ and $|\mathcal{M}^{++-}_{ggg}\rangle$. The other helicity amplitudes are obtained from $|\mathcal{M}^{+++}_{ggg}\rangle$ and $|\mathcal{M}^{++-}_{ggg}\rangle$ by the usual parity relation and by exploiting the symmetry of the gluons. Explicitly, choosing p_{i+1} as reference momentum for ϵ_{i,λ_i} we find,

$$\begin{aligned} |\mathcal{M}^{+++}_{ggg}\rangle &= \alpha\,\frac{1}{\sqrt{2}}\frac{M_H^4}{\langle p_1p_2\rangle\langle p_2p_3\rangle\langle p_3p_1\rangle}, \\ |\mathcal{M}^{++-}_{ggg}\rangle &= \beta\,\frac{1}{\sqrt{2}}\frac{[p_1p_2]^3}{[p_2p_3][p_1p_3]}, \end{aligned} \tag{4.10}$$

where the coefficients α and β are written in terms of the tensor coefficients,

$$\begin{aligned} \alpha &= \frac{s_{12}s_{13}s_{23}}{2M_H^4}\left(\frac{s_{12}}{s_{23}}A_{211} + \frac{s_{23}}{s_{13}}A_{232} - \frac{s_{13}}{s_{23}}A_{311} - 2A_{312}\right), \\ \beta &= \frac{s_{13}}{2}A_{211}. \end{aligned} \tag{4.11}$$

Likewise (3.14) yields the helicity amplitudes for $H \to q\bar{q}g$ in terms of spinor products. There is only one independent helicity amplitude $|\mathcal{M}_{q\bar{q}g}^{-++}\rangle$ and all other amplitudes can be obtained from $|\mathcal{M}_{q\bar{q}g}^{-++}\rangle$ using the usual parity and charge conjugation relations. By choosing p_1 as reference momentum for ϵ_{3,λ_3}, we obtain,

$$|\mathcal{M}_{q\bar{q}g}^{-++}\rangle = \gamma \frac{1}{\sqrt{2}} \frac{[p_2 p_3]^2}{[p_1 p_2]} . \tag{4.12}$$

The helicity coefficient γ is obtained from the tensor coefficients as,

$$\gamma = s_{12} \, A_1 \, . \tag{4.13}$$

As with the tensor coefficients, the helicity amplitude coefficients α, β and γ are vectors in colour space and have perturbative expansions,

$$\Omega = \lambda \sqrt{4\pi\alpha_s} T_\Omega \left[\Omega^{(0)} + \left(\frac{\alpha_s}{2\pi}\right) \Omega^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \Omega^{(2)} + \mathcal{O}(\alpha_s^3) \right] , \tag{4.14}$$

for $\Omega = \alpha, \beta, \gamma$. The colour factor is $T_\alpha = T_\beta = f^{a_1 a_2 a_3}$ and $T_\gamma = T^{a_3}_{i_1 j_2}$.

5. Calculation of the two-loop helicity coefficients

5.1 Calculation of two-loop Feynman amplitudes

The calculation of the two-loop Feynman amplitudes contributing to $H \to ggg$ and $H \to q\bar{q}g$ follows closely the calculation of the two-loop helicity amplitudes for $\gamma^* \to q\bar{q}g$ [29], which contribute to the NNLO corrections to $e^+e^- \to 3j$ and related event shapes [37,38], and of the two-loop helicity amplitudes for $q\bar{q} \to V\gamma$ [39]. We performed two completely independent calculations of the amplitudes, which provide a strong internal cross-check on the results.

The Feynman diagrams contributing to the i-loop amplitude $|\mathcal{M}^{(i)}\rangle$ $(i = 0, 1, 2)$ were all generated using QGRAF [40]. For $H \to ggg$, there are four diagrams at tree-level, 60 diagrams at one loop and 1306 diagrams at two loops, while for $H \to q\bar{q}g$, we have one diagram at tree-level, 15 diagrams at one loop and 228 diagrams at two loops. We use dimensional regularization [41–43] with $D = 4 - 2\epsilon$ dimensions. We therefore apply the D-dimensional projectors given in Eqs. (3.10) and (3.18) and perform the summation over colours and spins using computer algebra methods, mainly implemented in FORM [44]. When summing over the polarizations of the external gluons in the projectors, we use the axial gauge with a D-dimensional metric. Internal gluons are kept in Feynman gauge, resulting in internal ghost contributions to the loop amplitudes. The integrals appearing in the individual two-loop diagrams contain up to seven propagators in the denominator, and up to five irreducible scalar products in the numerator (i.e. scalar products which can not be expressed as linear combinations of the occurring propagators).

The reduction of the two-loop integrals to a small set of master integrals using integration-by-parts (IBP) [45,46] and Lorentz invariance (LI) [47] identities was performed using the Laporta algorithm [48], which is based on a lexicographic ordering of the integrals. We used two independent implementations of the Laporta algorithm: the MAPLE and FORM based implementation which was developed in the context of [29] and the recently developed C++ code REDUZE [49]. Both implementations are based on auxiliary topologies [29], and substantial work was required to automate the translation of the momentum assignments in the diagrams generated by QGRAF into the momentum sets of the auxiliary topologies. This process has been automated in FORM using an iterated shifting and matching algorithm for the momenta.

The two-loop master integrals relevant for this calculation are two-loop four-point functions with one leg off-shell. These functions were all computed in [50] in dimensional regularization. The results of [50] take the form of a Laurent series in ϵ, starting at ϵ^{-4}, with coefficients containing one- and two-dimensional harmonic polylogarithms (HPLs [51] and 2dHPLs [50]), which are a generalization of Nielsen's polylogarithms [52]. Several numerical implementations of HPLs and 2dHPLs are available [53].

Inserting the master integrals into the amplitudes and truncating the Laurent series to the required order, the unrenormalized one-loop and two-loop helicity coefficients are obtained. Their Laurent expansion contains HPLs and 2dHPLs up to weight 4. The expressions for the master integrals derived in [50] apply to the kinematical situation of a $1 \to 3$ decay, while the $H + 1j$ production corresponds to a $2 \to 2$ scattering process which is obtained from the decay kinematics by crossing. The crossing of the amplitudes requires the analytic continuation of the master integrals, which is described in detail in [54].

5.2 Ultraviolet renormalization

Renormalization of ultraviolet divergences is performed in the $\overline{\text{MS}}$ scheme. It is carried out by replacing the bare coupling α_0 with the renormalized coupling $\alpha_s \equiv \alpha_s(\mu^2)$, evaluated at the renormalization scale μ^2,

$$\alpha_0 \mu_0^{2\epsilon} S_\epsilon = \alpha_s \mu^{2\epsilon} \left[1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi} \right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left(\frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right] , \tag{5.1}$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma} \qquad \text{with Euler constant } \gamma = 0.5772\ldots$$

and μ_0^2 is the mass parameter introduced in dimensional regularization [41–43] to maintain a dimensionless coupling in the bare QCD Lagrangian density; β_0 and β_1 are the first two coefficients of the QCD β-function,

$$\beta_0 = \frac{11 C_A - 4 T_R N_F}{6}, \qquad \beta_1 = \frac{17 C_A^2 - 10 C_A T_R N_F - 6 C_F T_R N_F}{6} , \tag{5.2}$$

with the QCD colour factors,

$$C_A = N, \qquad C_F = \frac{N^2 - 1}{2N}, \qquad T_R = \frac{1}{2} . \tag{5.3}$$

The renormalization relation for the effective coupling λ is given in [55] as,

$$\lambda^U = \lambda \left[1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi} \right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{\epsilon} \right) \left(\frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right] . \tag{5.4}$$

We denote the i-loop contribution to the unrenormalized coefficients by $\Omega^{(i),\mathrm{U}}$, using the same normalization as for the decomposition of the renormalized amplitude (4.14). The renormalized coefficients are then obtained as,

$$\begin{aligned}
\Omega^{(0)} &= \Omega^{(0),\mathrm{U}}, \\
\Omega^{(1)} &= S_\epsilon^{-1}\Omega^{(1),\mathrm{U}} - \frac{3\beta_0}{2\epsilon}\Omega^{(0),\mathrm{U}}, \\
\Omega^{(2)} &= S_\epsilon^{-2}\Omega^{(2),\mathrm{U}} - \frac{5\beta_0}{2\epsilon}S_\epsilon^{-1}\Omega^{(1),\mathrm{U}} - \left(\frac{5\beta_1}{4\epsilon} - \frac{15\beta_0^2}{8\epsilon^2}\right)\Omega^{(0),\mathrm{U}}.
\end{aligned} \tag{5.5}$$

For the remainder of this paper we will set the renormalization scale $\mu^2 = M_H^2 = s_{123}$. The full scale dependence of the helicity coefficients are given by,

$$\begin{aligned}
\Omega = \lambda\sqrt{4\pi\alpha_s(\mu^2)}T_\Omega\Bigg\{&\Omega^{(0)} + \left(\frac{\alpha_s(\mu^2)}{2\pi}\right)\left[\Omega^{(1)} + \frac{3}{2}\beta_0\Omega^{(0)}\ln\left(\frac{\mu^2}{s_{123}}\right)\right] \\
&+ \left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2\left[\Omega^{(2)} + \left(\frac{5}{2}\beta_0\Omega^{(1)} + \frac{5}{2}\beta_1\Omega^{(0)}\right)\ln\left(\frac{\mu^2}{s_{123}}\right) + \frac{15}{8}\beta_0^2\Omega^{(0)}\ln^2\left(\frac{\mu^2}{s_{123}}\right)\right] \\
&+\mathcal{O}(\alpha_s^3)\Bigg\}.
\end{aligned} \tag{5.6}$$

5.3 Infrared factorization

The amplitudes contain infrared singularities that will be analytically canceled by those occurring in radiative processes of the same order (ultraviolet divergences are removed by renormalization). Catani [56] has shown how to organize the infrared pole structure of the one- and two-loop contributions renormalized in the $\overline{\mathrm{MS}}$ scheme in terms of the tree and renormalized one-loop amplitudes. This formula for the pole structure is proven [57] from the structure of soft and collinear radiation in perturbation theory and can be generalized to higher loop order.

The same factorization of pole terms applies to the helicity coefficients. In particular, the infrared behaviour of the one-loop coefficients is given by,

$$\Omega^{(1)} = \boldsymbol{I}_\Omega^{(1)}(\epsilon)\Omega^{(0)} + \Omega^{(1),finite}, \tag{5.7}$$

while the two-loop singularity structure is,

$$\begin{aligned}
\Omega^{(2)} = &\Bigg(-\frac{1}{2}\boldsymbol{I}_\Omega^{(1)}(\epsilon)\boldsymbol{I}_\Omega^{(1)}(\epsilon) - \frac{\beta_0}{\epsilon}\boldsymbol{I}_\Omega^{(1)}(\epsilon) \\
&+e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon} + K\right)\boldsymbol{I}_\Omega^{(1)}(2\epsilon) + \boldsymbol{H}_\Omega^{(2)}(\epsilon)\Bigg)\Omega^{(0)} \\
&+\boldsymbol{I}_\Omega^{(1)}(\epsilon)\Omega^{(1)} + \Omega^{(2),finite},
\end{aligned} \tag{5.8}$$

where the constant K is,

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{10}{9}T_R N_F. \tag{5.9}$$

For each of the processes under consideration, there is only one colour structure present at tree level. Adding higher loops does not introduce additional colour structures, and the amplitudes are therefore vectors in a one-dimensional space. Similarly, the infrared singularity operators $\boldsymbol{I}_\Omega^{(1)}(\epsilon)$ are 1×1 matrices in the colour space and are given by,

$$\boldsymbol{I}_\alpha^{(1)}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left[N\left(\frac{1}{\epsilon^2}+\frac{\beta_0}{N\epsilon}\right)(\mathtt{S}_{12}+\mathtt{S}_{13}+\mathtt{S}_{23})\right], \tag{5.10}$$

$$= \boldsymbol{I}_\beta^{(1)}(\epsilon), \tag{5.11}$$

$$\boldsymbol{I}_\gamma^{(1)}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left[N\left(\frac{1}{\epsilon^2}+\frac{3}{4\epsilon}+\frac{\beta_0}{2N\epsilon}\right)(\mathtt{S}_{13}+\mathtt{S}_{23})-\frac{1}{N}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\mathtt{S}_{12}\right], \tag{5.12}$$

where, since we have set $\mu^2 = s_{123}$,

$$\mathtt{S}_{ij} = \left(-\frac{s_{123}}{s_{ij}}\right)^\epsilon. \tag{5.13}$$

Note that on expanding $\mathtt{S}_{ij}$, imaginary parts are generated, the sign of which is fixed by the small imaginary part $+i0$ of s_{ij}. The origin of the various terms in Eqs. (5.10)–(5.12) is straightforward. Each parton pair ij in the event forms a radiating antenna of scale s_{ij}. Terms proportional to $\mathtt{S}_{ij}$ are canceled by real radiation emitted from leg i and absorbed by leg j. The soft singularities $\mathcal{O}(1/\epsilon^2)$ are independent of the identity of the participating partons and are universal. However, the collinear singularities depend on the identities of the participating partons. For each quark we find a contribution of $3/(4\epsilon)$ and for each gluon we find a contribution of $\beta_0/(2\epsilon)$ coming from the integral over the collinear splitting function.

Finally, the last term of Eq. (5.8) that involves $\boldsymbol{H}^{(2)}(\epsilon)$ produces only a single pole in ϵ and is given by,

$$\boldsymbol{H}_\Omega^{(2)}(\epsilon) = \frac{e^{\epsilon\gamma}}{4\,\epsilon\,\Gamma(1-\epsilon)}H_\Omega^{(2)}, \tag{5.14}$$

where the constant $H_\Omega^{(2)}$ is renormalization scheme dependent. As with the single pole parts of $\boldsymbol{I}_\Omega^{(1)}(\epsilon)$, the process-dependent $H_\Omega^{(2)}$ can be constructed by counting the number of radiating partons present in the event. In our case,

$$\begin{aligned} H_\alpha^{(2)} = H_\beta^{(2)} &= 3H_g^{(2)}, \\ H_\gamma^{(2)} &= 2H_q^{(2)} + H_g^{(2)}, \end{aligned} \tag{5.15}$$

where, in the $\overline{\mathrm{MS}}$ scheme,

$$\begin{aligned} H_q^{(2)} &= \left(\frac{7}{4}\zeta_3+\frac{409}{864}-\frac{11\pi^2}{96}\right)N^2+\left(-\frac{1}{4}\zeta_3-\frac{41}{108}-\frac{\pi^2}{96}\right)+\left(-\frac{3}{2}\zeta_3-\frac{3}{32}+\frac{\pi^2}{8}\right)\frac{1}{N^2} \\ &\quad+\left(\frac{\pi^2}{48}-\frac{25}{216}\right)\frac{(N^2-1)N_F}{N}, \end{aligned} \tag{5.16}$$

$$H_g^{(2)} = \left(\frac{1}{2}\zeta_3+\frac{5}{12}+\frac{11\pi^2}{144}\right)N^2+\frac{5}{27}N_F^2+\left(-\frac{\pi^2}{72}-\frac{89}{108}\right)NN_F-\frac{N_F}{4N}. \tag{5.17}$$

At leading order, one can insert the values of the tensorial coefficients given in Eqs. (3.12) and (3.20), into Eqs. (4.11) and (4.13) respectively to find,

$$\alpha^{(0)} = \beta^{(0)} = \gamma^{(0)} = 1\,. \tag{5.18}$$

The renormalized NLO helicity amplitude coefficients can be straightforwardly obtained to all orders in ϵ from the helicity coefficients $\Omega^{(1)}$. For practical purposes they are needed through to $\mathcal{O}(\epsilon^2)$ in evaluating the one-loop self-interference and the infrared divergent one-loop contribution to the two-loop amplitude, while only the finite piece is needed for the one-loop self-interference. They can be decomposed according to their colour structure as follows,

$$\Omega^{(1),finite} = \left(N\ A_\Omega^{(1)} + \frac{1}{N}\ B_\Omega^{(1)} + N_F C_\Omega^{(1)}\right). \tag{5.19}$$

The finite two-loop remainder is obtained by subtracting the predicted infrared structure (expanded through to $\mathcal{O}(\epsilon^0)$) from the renormalized helicity coefficient. We further decompose the finite remainder according to the colour casimirs as follows,

$$\Omega^{(2),finite} = \left(N^2 A_\Omega^{(2)} + N^0 B_\Omega^{(2)} + \frac{1}{N^2} C_\Omega^{(2)} + \frac{N_F}{N} D_\Omega^{(2)} + N N_F E_\Omega^{(2)} + N_F^2 F_\Omega^{(2)}\right). \tag{5.20}$$

All one- and two-loop coefficients are given in Appendix A and B respectively.

To calculate the two-loop contributions to Higgs-boson-plus-jet production at hadron colliders, the helicity amplitudes must be crossed to the appropriate kinematical situations. Two types crossings are required:

$$g(p_1) + g(p_2) \to H(p_4) + g(-p_3)\ , \qquad q(p_1) + \bar{q}(p_2) \to H(p_4) + g(-p_3)\ , \tag{5.21}$$

$$g(p_2) + g(p_3) \to H(p_4) + g(-p_1)\ , \qquad \bar{q}(p_2) + g(p_3) \to H(p_4) + \bar{q}(-p_1)\ . \tag{5.22}$$

The definitions of the helicity amplitudes in terms of momentum spinors (4.10,4.12) remain unchanged by the crossing, such that only the helicity coefficients α, β, γ are to be continued to the appropriate kinematical region. The analytical continuation of the polylogarithmic functions appearing in two-loop amplitudes is described in detail in [39,54]. We provide the one-loop and two-loop coefficients in all relevant analytic continuations in FORM format with the arXiv-submission of this article.

6. Conclusions

In this paper, we derived the two-loop corrections to the helicity amplitudes for the processes $H \to ggg$ and $H \to q\bar{q}g$. Our calculation was performed in dimensional regularization by applying D-dimensional projection operators to the most general tensor structure of the amplitude. Our results are expressed in terms of dimensionless helicity coefficients, which multiply the basic tree-level amplitudes, expressed in four-dimensional spinors. By applying Catani's infrared factorization formula, we extract the finite parts of the helicity coefficients, which are independent on the precise scheme used to define the helicity amplitudes. We provide compact analytic expressions for the two-loop helicity coefficients in terms of HPLs and 2dHPLs.

By crossing the Higgs boson to the final state and two partons to the initial state, these amplitudes describe the two-loop corrections to the parton-level process for $H+1j$ production and for the transverse momentum distribution of the Higgs boson. To compute the NNLO corrections to these processes, the newly derived two-loop amplitudes need to be combined with the previously known [18,19] crossings of the tree-level amplitudes for $H \to 5$ partons and one-loop amplitudes for $H \to 4$ partons into a parton-level event generator. The tree-level double-real radiation and one-loop real-virtual contributions both contain infrared singularities from soft or collinear real radiation. In order to numerically implement these contributions, the singular contributions must be subtracted and combined with infrared singularities from the two-loop integrals. Up to now, infrared subtraction at NNLO has not been fully accomplished for hadron collider processes involving final state jets. In the context of dijet production, a first proof-of-principle implementation of the double-real radiation contribution exists [26], and the infrared structure of the NNLO subtraction terms for hadron collider processes is largely understood [25]. With the amplitudes derived in this paper, it should thus become feasible to compute the NNLO corrections to $H \to 1j$ production and the Higgs boson transverse momentum distribution at hadron colliders.

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A. One-loop helicity coefficients

The finite contributions to the renormalized one-loop helicity coefficients, decomposed in colour factors according to (5.19) are:

$$\begin{aligned} A_\alpha^{(1)} = &\Bigg[\frac{1}{2}\Big(-G(1-z,0,y)-H(1,0,z)-G(0,1-z,y) \\ &-H(0,1,z)-H(0,z)G(1-z,y)+G(0,y)H(1,z)-G(0,y)H(0,z)\Big) \\ &+G(-z,1-z,y)+G(1,0,y)-H(1,z)G(-z,y) \\ &+\frac{11}{12}\Big(-G(1-z,y)+H(1,z)-H(0,z)-G(0,y)\Big)-\frac{\pi^2}{12}\Bigg] \\ &+\frac{1}{6}\Big((y+z)(1-y)-z^2\Big)+i\frac{11\pi}{4}\,, \end{aligned} \tag{A.1}$$

$$
\begin{aligned}
B_\alpha^{(1)} &= 0\,, && \text{(A.2)}\\
C_\alpha^{(1)} &= \frac{1}{6}\Big[G(1-z,y)-H(1,z)+H(0,z)+G(0,y)\Big]\\
&\quad -\frac{1}{6}\Big((y+z)(1-y)-z^2\Big)-i\frac{\pi}{2}\,, && \text{(A.3)}\\
A_\beta^{(1)} &= \Bigg[\frac{1}{2}\Big(-G(1-z,0,y)-H(1,0,z)-G(0,1-z,y)-H(0,1,z)\\
&\quad -H(0,z)G(1-z,y)+G(0,y)H(1,z)-G(0,y)H(0,z)\Big)+G(-z,1-z,y)\\
&\quad +G(1,0,y)-H(1,z)G(-z,y)\\
&\qquad +\frac{11}{12}\Big(-G(1-z,y)+H(1,z)-H(0,z)-G(0,y)\Big)\Bigg]\\
&\quad -\frac{z}{6(1-y-z)}\Big(1-\frac{1}{1-y-z}+\frac{z}{1-y-z}\Big)-\frac{\pi^2}{12}+i\frac{11\pi}{4}\,, && \text{(A.4)}\\
B_\beta^{(1)} &= 0\,, && \text{(A.5)}\\
C_\beta^{(1)} &= \frac{1}{6}\Bigg[G(1-z,y)-H(1,z)+H(0,z)+G(0,y)\\
&\quad -\frac{z}{1-y-z}\Big(-1+\frac{1}{1-y-z}-\frac{z}{1-y-z}\Big)\Bigg]-i\frac{\pi}{2}\,, && \text{(A.6)}\\
A_\gamma^{(1)} &= \frac{1}{6}\Bigg[3\Big(-G(0,y)H(1,z)+H(0,z)G(1-z,y)+H(0,1,z)+G(0,1-z,y)\\
&\quad -G(1,0,y)+G(1-z,0,y)\Big)+6\Big(H(1,z)G(-z,y)-G(-z,1-z,y)\Big)\\
&\quad +5\Big(G(0,y)+H(0,z)\Big)+\frac{13}{2}\Big(-H(1,z)+G(1-z,y)\Big)-\frac{89}{6}-\frac{3(z-1)}{2y}\Bigg]\\
&\quad -i\frac{11\pi}{4}\,, && \text{(A.7)}\\
B_\gamma^{(1)} &= \frac{1}{6}\Bigg[3\Big(-G(0,y)H(0,z)+G(1,0,y)-H(1,0,z)\Big)-\frac{\pi^2}{2}-\frac{27}{2}-\frac{3(z-1)}{2y}\Bigg]\,, && \text{(A.8)}\\
C_\gamma^{(1)} &= \frac{1}{6}\Bigg[\frac{1}{2}\Big(-G(0,y)-H(0,z)\Big)+2\Big(H(1,z)-G(1-z,y)\Big)+\frac{10}{3}\Bigg]+i\frac{\pi}{2}\,. && \text{(A.9)}
\end{aligned}
$$

B. Two-loop helicity coefficients

The finite contributions to the renormalized two-loop helicity coefficients, decomposed in colour factors according to (5.20) are:

$$
\begin{aligned}
A_\alpha^{(2)} = \Bigg[&\frac{1}{2}\Big(-G(1-z,-z,1-z,0,y)-G(1-z,-z,0,1-z,y)+G(1-z,1-z,0,0,y)\\
&+G(1-z,0,-z,1-z,y)+G(1-z,0,1-z,0,y)-G(1-z,0,1,0,y)\\
&+G(1-z,0,0,1-z,y)+H(1,1,0,0,z)+H(1,0,1,0,z)+H(1,0,0,1,z)\\
&+H(1,0,0,z)G(1-z,y)-H(1,0,z)G(-z,0,y)+H(1,0,z)G(1-z,-z,y)\\
&-G(1,0,y)H(1,0,z)+H(1,z)G(1-z,-z,0,y)+H(1,z)G(1-z,0,-z,y)
\end{aligned}
$$

$$
\begin{aligned}
&- H(1,z)G(1-z,0,0,y) - G(0,-z,1-z,y) + G(0,-z,y)H(1,0,z)\\
&- G(0,1-z,-z,1-z,y) + G(0,1-z,1-z,0,y) + G(0,1-z,0,1-z,y)\\
&+ G(0,1-z,-z,y)H(1,z) - G(0,-z,1-z,y) + G(0,1-z,1-z,y)H(0,z)\\
&- G(0,1-z,1,0,y) - G(0,1-z,0,y)H(1,z) - G(0,1-z,0,y)H(0,z)\\
&- H(0,z)G(1-z,0,1-z,y) - H(0,z)G(1-z,0,0,y) - G(0,y)H(1,1,0,z)\\
&- G(0,y)H(1,0,1,z) + G(0,y)H(1,0,0,z) - G(0,1,1-z,0,y) + H(0,1,1,0,z)\\
&- H(0,1,1,z)G(0,y) - G(0,1,0,1-z,y) + H(0,1,0,1,z)\\
&+ H(0,1,0,z)G(1-z,y) + H(0,1,0,z)G(0,y) + G(0,1,0,y)H(1,z)\\
&- G(0,1,0,y)H(0,z) - H(0,1,z)G(-z,0,y) - H(0,1,z)G(1,0,y)\\
&+ H(0,1,z)G(0,-z,y) + G(0,0,1-z,1-z,y) - G(0,0,1-z,y)H(1,z)\\
&- G(0,0,1-z,y)H(0,z) + H(0,0,1,1,z) + H(0,0,1,z)G(0,y)\\
&+ H(0,0,z)G(1-z,1-z,y) + H(0,0,z)G(1-z,0,y) + H(0,0,z)G(0,1-z,y)\\
&+ G(0,0,y)H(1,1,z) - G(0,0,y)H(1,0,z) - G(0,0,y)H(0,1,z)\\
&+ G(0,0,y)H(0,0,z) + H(0,z)G(-z,1-z,0,y) + H(0,z)G(-z,0,1-z,y)\\
&- H(0,z)G(1-z,-z,1-z,y) - H(0,z)G(1-z,1-z,0,y)\\
&- H(0,z)G(1-z,1,0,y) + H(0,z)G(1,1-z,0,y) + H(0,z)G(1,0,1-z,y)\Big)\\
&\qquad +\Big(- G(-z,1-z,1-z,0,y) - G(-z,1-z,0,1-z,y)\\
&- G(-z,0,1-z,1-z,y) - G(1-z,1-z,-z,1-z,y) + G(1-z,1-z,1,0,y)\\
&+ G(1-z,1,1-z,0,y) + G(1-z,1,0,1-z,y) - G(1-z,1,0,0,y)\\
&- G(1-z,0,-z,1-z,y) - G(1,1-z,0,0,y) - H(1,1,0,z)G(-z,y)\\
&- H(1,1,z)G(-z,0,y) - G(1,0,1-z,0,y) + H(1,0,1,z)G(-z,y)\\
&- G(1,0,0,1-z,y) + G(1,0,0,y)H(1,z) - H(1,0,z) + H(1,0,z)G(-z,1-z,y)\\
&- H(1,0,z)G(1-z,1-z,y) + H(1,0,z)G(1-z,0,y) + H(1,z)G(-z,1-z,0,y)\\
&+ H(1,z)G(-z,0,1-z,y) + H(1,z)G(1-z,1-z,-z,y)\\
&- H(1,z)G(1-z,1,0,y) - G(0,-z,1-z,1-z,y) + G(0,-z,1-z,0,y)\\
&+ G(0,-z,0,1-z,y) + G(0,-z,1-z,y) - G(0,-z,y)H(1,1,z)\\
&+ G(0,-z,1-z,y)H(0,z) - G(0,-z,0,y)H(1,z) + G(0,-z,1-z,y)H(1,z)\\
&+ H(0,1,1,z)G(-z,y) - H(0,1,z)G(-z,1-z,y) + H(0,1,z)G(1-z,1-z,y)\\
&+ H(0,1,z)G(1-z,0,y) + G(0,0,-z,1-z,y) - G(0,0,-z,y)H(1,z)\\
&- G(0,0,1,0,y) - H(0,z)G(-z,1-z,1-z,y) + H(0,z)G(1-z,0,0,y)\\
&- H(0,z)G(1,0,0,y)\Big)\\
&\qquad +\frac{3}{2}\Big(- G(0,-z,1-z,y)H(0,z) - H(0,1,z)G(1-z,-z,y)\\
&- H(0,0,1,z)G(1-z,y)\Big)\\
&\qquad +2\Big(- G(-z,-z,-z,1-z,y) + G(-z,-z,1-z,1-z,y)
\end{aligned}
$$

$$
\begin{aligned}
&+G(-z,1-z,-z,1-z,y)+G(1-z,-z,-z,1-z,y)-G(1,1,1,0,y)\\
&+G(1,1,0,0,y)+H(1,1,0,z)G(1-z,y)+H(1,1,z)G(-z,-z,y)+G(1,0,1,0,y)\\
&+H(1,z)G(-z,-z,-z,y)-H(1,z)G(-z,-z,1-z,y)\\
&-H(1,z)G(-z,1-z,-z,y)-H(1,z)G(1-z,-z,-z,y)+G(0,1,1,0,y)\\
&+H(0,1,z)G(-z,-z,y)+H(0,0,1,z)G(-z,y)\Big)\\
&\qquad+\frac{11}{24}\Big(-G(0,1-z,y)H(0,z)+H(0,1,z)G(0,y)-H(0,z)G(1-z,0,y)\\
&+G(0,y)H(1,0,z)\Big)\\
&\qquad+\frac{11}{6}\Big(-G(-z,-z,1-z,y)+G(1-z,1-z,0,y)+G(1-z,0,1-z,y)\\
&+G(1-z,0,0,y)-G(1,1,0,y)+H(1,0,0,z)-H(1,0,z)G(1-z,y)\\
&+H(1,z)G(-z,-z,y)-H(1,z)G(1-z,0,y)+G(0,1-z,1-z,y)\\
&-G(0,1-z,y)H(1,z)+G(0,1-z,0,y)-H(0,1,1,z)+H(0,1,z)G(-z,y)\\
&+G(0,0,1-z,y)+H(0,0,z)G(1-z,y)+H(0,0,z)G(0,y)-G(0,0,y)H(1,z)\\
&+G(0,0,y)H(0,z)+H(0,z)G(1-z,1-z,y)+G(0,y)H(1,1,z)\Big)\\
&\qquad+\frac{11}{4}\Big(-H(1,0,1,z)+H(0,1,z)G(1-z,y)+H(0,1,0,z)\Big)\\
&\qquad+\frac{11}{3}\Big(-G(1,0,0,y)-G(-z,1-z,1-z,y)-H(1,1,z)G(-z,y)\\
&+H(1,z)G(-z,1-z,y)\Big)\\
&\qquad+\frac{55}{12}\Big(-G(1-z,-z,1-z,y)+H(1,z)G(1-z,-z,y)-G(0,1,0,y)\Big)\\
&\qquad+\frac{\pi^2}{3}\Big(G(1,1,y)-G(0,1,y)+H(1,1,z)-G(1,0,y)+H(0,0,z)+G(0,0,y)\Big)\\
&\qquad+\Big(\frac{\zeta_3}{4}-\frac{33\pi^2}{8}\Big)\Big(-G(1-z,y)+H(1,z)-H(0,z)-G(0,y)\Big)\\
&\qquad+\Big(-\frac{121}{48}+\frac{\pi^2}{3}\Big)\Big(-G(1-z,1-z,y)-H(1,1,z)+H(1,z)G(1-z,y)\\
&-H(0,0,z)-G(0,0,y)\Big)\\
&\qquad+\Big(\frac{49}{48}-\frac{\pi^2}{8}\Big)\Big(-G(1-z,0,y)-G(0,1-z,y)-H(0,z)G(1-z,y)\\
&+G(0,y)H(1,z)-G(0,y)H(0,z)\Big)\\
&\qquad+\Big(\frac{67}{18}+\frac{\pi^2}{12}\Big)\Big(G(-z,1-z,y)+G(1,0,y)-H(1,z)G(-z,y)\Big)\\
&-\Big(\frac{245}{144}-\frac{\pi^2}{8}\Big)H(1,0,z)-\Big(\frac{389}{144}-\frac{\pi^2}{8}\Big)H(0,1,z)\\
&-\Big(\frac{13}{8}+\frac{451\pi^2}{96}\Big)G(1-z,y)+\Big(\frac{13}{8}+\frac{1265\pi^2}{288}\Big)H(1,z)\\
&+\Big(\frac{13}{8}+\frac{1133\pi^2}{288}\Big)\Big(-H(0,z)-G(0,y)\Big)+\frac{11\pi^2G(1,y)}{36}
\end{aligned}
$$

$$-\frac{1}{36}\Big(\frac{5029\pi^2}{24}-72\zeta_4+\frac{99\zeta_3}{4}+\frac{3\pi^4}{16}-\frac{1321}{6}\Big)\Big]$$

$$+\frac{1}{6}\Big((y+z)(1-y)-z^2\Big)\Big[+G(1,0,y)+G(-z,1-z,y)-H(1,z)G(-z,y)$$

$$+G(0,y)H(1,z)-G(0,y)H(0,z)$$

$$+\frac{1}{2}\Big(-G(1-z,0,y)-H(1,0,z)-G(0,1-z,y)-H(0,1,z)$$

$$-H(0,z)G(1-z,y)-G(0,y)H(1,z)+G(0,y)H(0,z)\Big)$$

$$-\frac{41}{12}\Big(G(1-z,y)-H(1,z)\Big)+\frac{19}{12}H(0,z)-\frac{3\pi^2}{2}+\frac{247}{18}\Big]$$

$$+\Big(\frac{25z}{12}(-1+\frac{1}{1-y-z})-\frac{15z^2}{4(1-y-z)}-\frac{yz}{6}+\frac{5z^2}{6}(1+\frac{2z}{1-y-z}$$

$$+\frac{1}{(1-y-z)^2}(1-2z+z^2))\Big)\Big[G(0,y)H(0,z)+H(1,0,z)-G(1,0,y)\Big]$$

$$+\Big(\frac{25y}{12z}-\frac{9y}{4}+\frac{yz}{6}-\frac{15y^2}{4z}+y^2+\frac{5y^2}{6}(\frac{2y}{z}+\frac{1}{z^2}(1-2y+y^2))\Big)\times$$

$$\Big[G(1-z,0,y)-G(1,0,y)-G(-z,1-z,y)+H(1,z)G(-z,y)+G(0,1-z,y)$$

$$+H(0,1,z)-G(0,y)H(1,z)\Big]$$

$$+\Big(\frac{25z}{12y}-\frac{9z}{4}+\frac{yz}{6}-\frac{15z^2}{4y}+z^2+\frac{5z^2}{6}(\frac{2z}{y}+\frac{1}{y^2}(1-2z+z^2))\Big)\times$$

$$\Big[H(0,z)G(1-z,y)-G(-z,1-z,y)+H(1,z)G(-z,y)\Big]$$

$$+\frac{1}{36}\Big(63-93(y+z)+4yz+\frac{30z}{y}(1-2z+z^2)+\frac{30y}{z}(1-2y+y^2)$$

$$+30(y^2+z^2)\Big)\Big[G(1-z,y)-H(1,z)\Big]$$

$$-\frac{1}{36}\Big(-63z+60z^2-30z^2(1-z)(\frac{1}{y}+\frac{1}{1-y-z})+26y(1-y-z)\Big)H(0,z)$$

$$-\frac{1}{36}\Big(\frac{93z(1-z)}{2}-\frac{145y}{2}+\frac{27yz}{2}+\frac{79y^2}{2}+\frac{30z}{1-y-z}(-1+2z-z^2)$$

$$-\frac{30y^2(1-y)}{z}\Big)G(0,y)$$

$$+\frac{\pi^2}{2}\Big(\frac{25z}{36(1-y-z)}-\frac{2z}{9}+\frac{5z^2}{18(1-y-z)}(2z-\frac{9}{2}+\frac{1}{1-y-z}(1-2z+z^2))$$

$$-\frac{7z^2}{36}-\frac{19yz}{36}+\frac{17y(1-y)}{36}\Big)$$

$$+i\pi\Big[\frac{55}{24}\Big(-H(0,z)G(1-z,y)-H(0,z)G(0,y)-H(0,1,z)$$

$$-2H(1,z)G(-z,y)+H(1,z)G(0,y)-H(1,0,z)-G(1-z,0,y)$$

$$+2G(-z,1-z,y)-G(0,1-z,y)+2G(1,0,y)+\frac{11}{6}(-H(0,z)+H(1,z)$$

$$- G(1-z,y) - G(0,y)) + \frac{1}{3}(y(1-y-z)+z(1-z))\Big) - \frac{77\pi^2}{288} + \frac{3\zeta_3}{4} + \frac{185}{24}\Big] , \quad \text{(B.1)}$$

$$B_\alpha^{(2)} = 0\,, \quad \text{(B.2)}$$

$$C_\alpha^{(2)} = 0\,, \quad \text{(B.3)}$$

$$\begin{aligned}
D_\alpha^{(2)} = \; & \frac{y}{12}\Big(\frac{y}{z^2}(-1+2y-y^2) + \frac{2}{z}(1-y^2) - 4 + 2z + y\Big)\Big[+ G(1-z,0,y) \\
& + G(0,1-z,y) + H(0,1,z) - G(0,y)H(1,z) - G(1,0,y) + H(1,z)G(-z,y) \\
& - G(-z,1-z,y)\Big] \\
& + \frac{z}{12}\Big(\frac{z}{(1-y-z)^2}(-1+2z-z^2) + \frac{2}{1-y-z}(1-z^2) - 2 - z - 2y\Big) \times \\
& \Big[H(1,0,z) + G(0,y)H(0,z) - G(1,0,y)\Big] \\
& + \frac{z}{12}\Big(\frac{z}{y^2}(-1+2z-z^2) + \frac{2}{y}(1-z^2) - 4 + z + 2y\Big)\Big[H(0,z)G(1-z,y) \\
& + H(1,z)G(-z,y) - G(-z,1-z,y)\Big] \\
& + \frac{1}{36}\Big(\frac{15}{2} + \frac{3z}{y}(-1+2z-z^2) + \frac{3y}{z}(-1+2y-y^2) - \frac{9}{2}(y^2+z^2) \\
& - 3(y+z+yz)\Big)\Big[G(1-z,y) - H(1,z)\Big] \\
& - \frac{1}{36}\Big(3z^2(1-z)(\frac{1}{y} + \frac{1}{1-y-z}) - 6z - \frac{15z^2}{2}\Big)H(0,z) \\
& - \frac{1}{36}\Big(\frac{3z}{1-y-z}(1-2z+z^2) - 3z(1-z) + \frac{3y^2(1-y)}{z} \\
& - 6y - 3yz - \frac{9y^2}{2}\Big)G(0,y) - \frac{1}{18}\Big[-\frac{201}{8} + 18\zeta_3\Big] - \frac{1}{6}\Big((y+z)(1-y) - z^2\Big) \\
& + \frac{\pi^2}{72}\Big(\frac{z}{1-y-z}(2 - \frac{z}{1-y-z} + \frac{2z^2}{1-y-z} - 2z^2 - \frac{z^3}{1-y-z}) - z(2+z+2y)\Big) \\
& + i\frac{\pi}{4}\,, \qquad \text{(B.4)}
\end{aligned}$$

$$\begin{aligned}
E_\alpha^{(2)} = \; & \frac{1}{6}\Big[\frac{1}{2}\Big(+ G(0,1-z,y)H(0,z) - H(0,1,z)G(0,y) + H(0,z)G(1-z,0,y) \\
& - G(0,y)H(1,0,z)\Big) \\
& + 2\Big(G(-z,-z,1-z,y) - G(1-z,1-z,0,y) - G(1-z,0,1-z,y) \\
& - G(1-z,0,0,y) - G(0,1-z,0,y) + G(1,1,0,y) - H(1,0,0,z) \\
& + H(1,0,z)G(1-z,y) - H(1,z)G(-z,-z,y) + H(1,z)G(1-z,0,y) \\
& - G(0,1-z,1-z,y) + G(0,1-z,y)H(1,z) + H(0,1,1,z) - H(0,1,z)G(0,y) \\
& - H(0,1,z)G(-z,y) + H(0,1,z)G(0,y) - G(0,0,1-z,y) \\
& - H(0,0,z)G(1-z,y) - H(0,0,z)G(0,y) + G(0,0,y)H(1,z) - G(0,0,y)H(0,z)
\end{aligned}$$

$$
\begin{aligned}
&- H(0,z)G(1-z,1-z,y) - G(0,y)H(1,1,z)\Big) \\
&\quad + 3\Big(H(1,0,1,z) + G(0,1-z,y) - H(0,1,0,z) - H(0,1,z)G(1-z,y)\Big) \\
&\quad + 4\Big(G(-z,1-z,1-z,y) + H(1,1,z)G(-z,y) + G(1,0,0,y) \\
&- H(1,z)G(-z,1-z,y)\Big) \\
&\quad + 5\Big(G(1-z,-z,1-z,y) - H(1,z)G(1-z,-z,y) + G(0,1,0,y)\Big) \\
&\quad - \frac{10}{3}\Big(G(-z,1-z,y) + G(1,0,y) - H(1,z)G(-z,y)\Big) \\
&\quad + \frac{11}{2}\Big(- G(1-z,1-z,y) - H(1,1,z) + H(1,z)G(1-z,y) - H(0,0,z) \\
&- G(0,0,y)\Big) - \frac{7}{2}\Big(- H(1,0,z) - H(0,1,z)\Big) - \frac{19G(0,1-z,y)}{6} \\
&\quad + \frac{1}{6}\Big(- G(1-z,0,y) - H(0,z)G(1-z,y) + G(0,y)H(1,z) - G(0,y)H(0,z)\Big) \\
&\quad + \Big(\frac{103}{18} + \frac{41\pi^2}{8}\Big)G(1-z,y) - \Big(\frac{103}{18} + \frac{7\pi^2}{24}\Big)H(1,z) + \Big(\frac{35}{36} - \frac{5\pi^2}{24}\Big)H(0,z) \\
&+ \Big(\frac{103}{18} - \frac{5\pi^2}{24}\Big)G(0,y) - \frac{\pi^2}{3}\Big(G(1,y) + \frac{27G(1-z,y)}{2}\Big) - \frac{1}{6}\Big(\frac{1781}{12} + \frac{63\zeta_3}{2} \\
&- \frac{1879\pi^2}{24}\Big)\Big] \\
&\quad + \frac{1}{36}\Big((y+z)(1-y) - z^2\Big)\Big[6\Big(- G(-z,1-z,y) - G(1,0,y) \\
&+ H(1,z)G(-z,y)\Big) \\
&\quad + 3\Big(- G(0,y)H(1,z) + G(1-z,0,y) + H(1,0,z) + G(0,1-z,y) \\
&+ H(0,1,z) + H(0,z)G(1-z,y)\Big) + G(0,y)H(0,z) + 2G(0,y)H(0,z) \\
&+ 23G(1-z,y) - 23H(1,z) - 19H(0,z) + \frac{\pi^2}{2} - \frac{275}{3}\Big] \\
&\quad + \Big(\frac{-7y}{6}(1-\frac{1}{z}) + \frac{11y^3}{12z^2}(\frac{z^2}{y} + \frac{1}{y} - 2 + 2z + y) - \frac{3y^2}{z}\Big)\Big[- G(1-z,0,y) \\
&+ G(-z,1-z,y) + G(1,0,y) - H(1,z)G(-z,y) - G(0,1-z,y) \\
&- H(0,1,z) + G(0,y)H(1,z)\Big] \\
&\quad + \Big(\frac{-7z}{6}(1-\frac{1}{y}) + \frac{11z^3}{12y^2}(\frac{y^2}{z} + \frac{1}{z} - 2 + 2y + z) - \frac{3z^2}{y}\Big)\Big[G(-z,1-z,y) \\
&- H(1,z)G(-z,y) - H(0,z)G(1-z,y)\Big] \\
&\quad + \Big(\frac{-7z}{6}(1-\frac{1}{1-y-z}) + \frac{11z^3}{12(1-y-z)^2}(\frac{(1-y-z)^2}{z} + \frac{1}{z} - 2 \\
&+ 2(1-y-z) + z) - \frac{3z^2}{1-y-z}\Big)\Big[- H(1,0,z) + G(1,0,y) - G(0,y)H(0,z)\Big]
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{1}{36}\Big(- \frac{57}{2} + \frac{33z}{y}(- 1 + 2z - z^2 - yz) + \frac{33y}{z}(- 1 + 2y - y^2 - yz) \\
& + \frac{123}{2}(y+z) - \frac{7yz}{2}\Big)\Big[G(1-z,y) - H(1,z)\Big] \\
& - \frac{1}{36}\Big(- \frac{57}{2}(1+2z^2) + \frac{39z}{2} + 33z^2(1-z)(\frac{1}{y} + \frac{1}{1-y-z}) \\
& + \frac{77y}{2}(- 1 + y + z)\Big)H(0,z) \\
& - \frac{1}{36}\Big(\frac{33z}{1-y-z}(1-2z+z^2) + \frac{105z^2}{2}(1-\frac{1}{z}) + y(\frac{77}{2} - \frac{27z}{2} + \frac{33y}{z} \\
& - 43y - \frac{33y^2}{z})\Big)G(0,y) \\
& + \frac{\pi^2}{2}\Big(\frac{7z}{18}(1 - \frac{1}{1-y-z}) + \frac{11z^2}{36(1-y-z)^2}(- 1 - (1-y-z)^2 + 2z \\
& - 2z(1-y-z) - z^2) + \frac{z^2}{1-y-z}\Big) \\
& + i\pi\Big[\frac{5}{12}\Big(H(0,z)G(1-z,y) + H(0,z)G(0,y) + H(0,1,z) \\
& + 2H(1,z)G(-z,y) - H(1,z) * G(0,y) + H(1,0,z) + G(1-z,0,y) \\
& - 2G(-z,1-z,y) + G(0,1-z,y) - 2G(1,0,y) + \frac{11}{3}(+ H(0,z) - H(1,z) \\
& + G(1-z,y) + G(0,y))\Big) + \frac{65}{72}\Big(z(z-1) - y(1-y-z)\Big) + \frac{7\pi^2}{144} - \frac{71}{18}\Big], \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
F_\alpha^{(2)} = \; & \frac{1}{36}\Big[3\Big(G(1-z,1-z,y) + H(1,1,z) - H(1,z)G(1-z,y) + H(0,0,z) \\
& + G(0,0,y)\Big) + G(1-z,0,y) - H(1,0,z) + G(0,1-z,y) - H(0,1,z) \\
& + H(0,z)G(1-z,y) - G(0,y)H(1,z) + G(0,y)H(0,z) \\
& + \frac{10}{3}\Big(- G(0,y) - H(0,z) + H(1,z) - G(1-z,y)\Big) - \frac{29\pi^2}{4}\Big] \\
& + \frac{1}{36}\Big((y+z)(1-y) - z^2\Big)\Big[- G(0,y) - H(0,z) + H(1,z) \\
& - G(1-z,y) + \frac{10}{3}\Big] - \frac{z(1-y-z)}{18}G(0,y) - \frac{y(1-y-z)}{18}H(0,z) \\
& - \frac{yz}{18}\Big(G(1-z,y) - H(1,z)\Big) \\
& + i\frac{5\pi}{36}\Big[- H(0,z) + H(1,z) - G(1-z,y) - G(0,y) \\
& + \Big(y(1-z-y) + 2 + z - z^2\Big)\Big], \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
A_\beta^{(2)} = \; & \frac{1}{2}\Big[- G(1-z,-z,1-z,0,y) - G(1-z,-z,0,1-z,y) + G(1-z,1-z,0,0,y) \\
& - G(1-z,0,-z,1-z,y) + G(1-z,0,1-z,0,y) - G(1-z,0,1,0,y) \\
& + G(1-z,0,0,1-z,y) + H(1,1,0,0,z) + H(1,0,1,0,z) + H(1,0,0,1,z)
\end{aligned}
$$

$$
\begin{aligned}
&+H(1,0,0,z)G(1-z,y)-H(1,0,z)G(-z,0,y)+H(1,0,z)G(1-z,-z,y)\\
&-G(1,0,y)H(1,0,z)+H(1,z)G(1-z,-z,0,y)+H(1,z)G(1-z,0,-z,y)\\
&-H(1,z)G(1-z,0,0,y)-G(0,-z,1-z,y)H(0,z)+G(0,-z,y)H(1,0,z)\\
&-G(0,1-z,-z,1-z,y)+G(0,1-z,-z,y)H(1,z)+G(0,1-z,1-z,y)H(0,z)\\
&+G(0,1-z,0,1-z,y)-G(0,1-z,1,0,y)+G(0,1-z,1-z,0,y)\\
&-G(0,1-z,0,y)H(1,z)-G(0,1-z,0,y)H(0,z)-G(0,1,1-z,0,y)\\
&+H(0,1,1,0,z)-H(0,1,1,z)G(0,y)-G(0,1,0,1-z,y)+H(0,1,0,1,z)\\
&+H(0,1,0,z)G(1-z,y)+H(0,1,0,z)G(0,y)+G(0,1,0,y)H(1,z)\\
&-G(0,1,0,y)H(0,z)-H(0,1,z)G(-z,0,y)-H(0,1,z)G(1,0,y)\\
&+H(0,1,z)G(0,-z,y)+H(0,z)G(-z,1-z,0,y)+H(0,z)G(-z,0,1-z,y)\\
&-H(0,z)G(1-z,-z,1-z,y)-H(0,z)G(1-z,1,0,y)+H(0,z)G(1,1-z,0,y)\\
&+H(0,z)G(1,0,1-z,y)-H(0,z)G(1-z,1-z,0,y)\\
&-H(0,z)G(1-z,0,1-z,y)+H(0,z)G(1-z,0,0,y)-G(0,y)H(1,1,0,z)\\
&-G(0,y)H(1,0,1,z)+G(0,y)H(1,0,0,z)+G(0,0,1-z,1-z,y)\\
&-G(0,0,1-z,y)H(1,z)-G(0,0,1-z,y)H(0,z)+H(0,0,1,1,z)\\
&+H(0,0,1,z)G(0,y)+H(0,0,z)G(1-z,1-z,y)+H(0,0,z)G(1-z,0,y)\\
&+H(0,0,z)G(0,1-z,y)-G(0,0,y)H(1,0,z)+G(0,0,y)H(1,1,z)\\
&-G(0,0,y)H(0,1,z)+G(0,0,y)H(0,0,z)\\
&\qquad+2\Big(-G(-z,1-z,1-z,0,y)-G(-z,1-z,0,1-z,y)\\
&-G(-z,0,1-z,1-z,y)-G(1-z,1-z,-z,1-z,y)+G(1-z,1-z,1,0,y)\\
&+G(1-z,1,1-z,0,y)+G(1-z,1,0,1-z,y)-G(1-z,1,0,0,y)\\
&-G(1,1-z,0,0,y)-H(1,1,0,z)G(-z,y)-H(1,1,z)G(-z,0,y)\\
&-G(1,0,1-z,0,y)+H(1,0,1,z)G(-z,y)-H(1,0,z)G(1-z,1-z,y)\\
&+H(1,0,z)G(1-z,0,y)-G(1,0,0,1-z,y)+G(1,0,0,y)H(1,z)\\
&+H(1,0,z)G(-z,1-z,y)+H(1,z)G(-z,1-z,0,y)+H(1,z)G(-z,0,1-z,y)\\
&+H(1,z)G(1-z,1-z,-z,y)-H(1,z)G(1-z,1,0,y)-G(0,-z,1-z,1-z,y)\\
&+G(0,-z,1-z,0,y)+G(0,-z,0,1-z,y)-G(0,-z,0,y)H(1,z)\\
&+G(0,-z,1-z,y)H(1,z)-G(0,-z,y)H(1,1,z)+H(0,1,1,z)G(-z,y)\\
&-H(0,1,z)G(-z,1-z,y)+H(0,1,z)G(1-z,1-z,y)+H(0,1,z)G(1-z,0,y)\\
&+G(0,0,-z,1-z,y)-G(0,0,-z,y)H(1,z)-H(0,z)G(-z,1-z,1-z,y)\\
&-H(0,z)G(1,0,0,y)-G(0,0,1,0,y)\Big)\\
&\qquad+3\Big(-H(0,1,z)G(1-z,-z,y)-H(0,0,1,z)G(1-z,y)\Big)\\
&\qquad+4\Big(-G(-z,-z,-z,1-z,y)+G(-z,-z,1-z,1-z,y)\\
&+G(-z,1-z,-z,1-z,y)+G(1-z,-z,-z,1-z,y)-G(1,1,1,0,y)\\
&+G(1,1,0,0,y)+H(1,1,0,z)G(1-z,y)+H(1,1,z)G(-z,-z,y)+G(1,0,1,0,y)
\end{aligned}
$$

$$
\begin{aligned}
&+H(1,z)G(-z,-z,-z,y)-H(1,z)G(-z,-z,1-z,y)\\
&-H(1,z)G(-z,1-z,-z,y)-H(1,z)G(1-z,-z,-z,y)+G(0,1,1,0,y)\\
&+H(0,1,z)G(-z,-z,y)+H(0,0,1,z)G(-z,y)\Big)\Big]\\
&\quad+\frac{11}{12}\Big[-\frac{\pi^2G(0,1,y)}{3}+\frac{1}{2}\Big(-G(0,1-z,y)H(0,z)+H(0,1,z)G(0,y)\\
&-H(0,z)G(1-z,0,y)\Big)-G(1-z,1,0,y)-G(1,1-z,0,y)-H(1,1,0,z)\\
&-G(1,0,1-z,y)+G(1,0,y)H(1,z)-H(1,0,z)G(1-z,y)\\
&+H(0,1,0,z)+H(0,1,z)G(1-z,y)\\
&\quad+2\Big(-G(-z,-z,1-z,y)+G(1-z,1-z,0,y)+G(1-z,0,1-z,y)\\
&+G(1-z,0,0,y)-H(1,0,1,z)+H(1,0,0,z)+H(1,z)G(-z,-z,y)\\
&-H(1,z)G(1-z,0,y)+G(0,1-z,1-z,y)+G(0,1-z,0,y)-H(0,1,1,z)\\
&+H(0,1,z)G(-z,y)+H(0,z)G(1-z,1-z,y)+G(0,y)H(1,1,z)\\
&+G(0,0,1-z,y)+H(0,0,z)G(1-z,y)+H(0,0,z)G(0,y)-G(0,0,y)H(1,z)\\
&+G(0,0,y)H(0,z)-G(0,1-z,y)H(1,z)\Big)\\
&\quad+3\Big(-G(1-z,-z,1-z,y)+H(1,z)G(1-z,-z,y)\Big)\\
&\quad+4\Big(-G(-z,1-z,1-z,y)-H(1,1,z)G(-z,y)-G(1,0,0,y)\\
&+H(1,z)G(-z,1-z,y)\Big)\Big]\\
&\quad+\frac{1}{36}\Big[33\Big(3H(1,1,0,z)+2H(0,1,0,z)-4G(0,1,0,y)+\frac{3G(0,y)H(1,0,z)}{2}\Big)\\
&\quad+134\Big(G(-z,1-z,y)+G(1,0,y)-H(1,z)G(-z,y)\Big)\\
&\quad+\frac{9}{2}\Big(G(1-z,y)-H(1,z)\Big)+\frac{389}{4}\Big(-H(1,0,z)-H(0,1,z)\Big)\\
&\quad-\frac{147H(0,z)G(1-z,y)}{4}-\frac{231G(0,y)}{2}-\frac{333H(0,z)}{2}+\frac{943}{6}\Big]\\
&\quad+\frac{\pi^2}{36}\Big[12\Big(H(1,1,z)+H(0,0,z)+G(0,0,y)\Big)+\frac{G(1,1,y)}{12}-G(0,1,y)\\
&+\frac{143G(1,1,y)}{12}+3G(-z,1-z,y)+\frac{11G(1-z,y)}{8}+\frac{9H(1,0,z)}{2}-9G(1,0,y)\\
&+\frac{77H(1,z)}{8}-3H(1,z)G(-z,y)+\frac{9H(0,1,z)}{2}+\frac{9H(0,z)G(1-z,y)}{2}+\frac{99G(0,y)}{8}\\
&+\frac{55H(0,z)}{8}-\frac{5029}{24}-\frac{3\pi^2}{16}\Big]\\
&\quad-\frac{\zeta_3}{4}\Big[G(1-z,y)-H(1,z)+G(0,y)+H(0,z)+\frac{121}{12}\Big]+2\zeta_4\\
&\quad+\frac{1}{36}\Big(-\frac{363}{4}+12\pi^2\Big)\Big[-G(1-z,1-z,y)+H(1,z)G(1-z,y)-H(0,0,z)
\end{aligned}
$$

$$
\begin{aligned}
&- G(0,0,y) - H(1,1,z)\Big] \\
&\qquad + \frac{1}{36}\Big(- \frac{147}{4} + \frac{9\pi^2}{2} - \frac{3zy}{(1-y-z)^2} + \frac{75y}{z} + \frac{30y^2}{z^2}\Big)\Big[G(1-z,0,y) \\
&+ G(0,1-z,y) - G(0,y)H(1,z) + G(0,y)H(0,z)\Big] \\
&\qquad + \frac{z}{36(1-y-z)^3}\Big(12(1-y-z)^2 + 3z(1-y-z) + 2z^2\Big)\Big[6(H(1,1,0,z) \\
&- G(1,1,0,y)) + 3(H(0,z)G(1,0,y) + H(0,1,0,z) + G(0,1,0,y) \\
&+ G(0,y)H(1,0,z)) + \frac{\pi^2}{2}(- H(0,z) + G(0,y) + 2H(1,z) + 2G(1,y))\Big] \\
&\qquad + \frac{1}{36(1-y-z)^3}\Big(2 - 3(1-y-z) + 12(1-y-z)^2\Big)\Big[- 6H(1,1,0,z) \\
&- 3G(0,1,0,y) - 3G(0,y)H(1,0,z) - \pi^2 H(1,z) - \frac{\pi^2}{2}G(0,y) + 6\zeta_3\Big] \\
&\qquad + \frac{zy}{36(1-y-z)^2}\Big[6\Big(G(-z,1-z,y) - H(1,z)G(-z,y) + \frac{\pi^2 H(1,z)}{1-y-z} \\
&- \frac{H(1,0,z)}{z} - \frac{G(1,0,y)}{z}\Big) - \frac{33}{2}\Big(G(1-z,y) - H(1,z)\Big) + \frac{18}{1-y-z}\Big(2H(1,1,0,z) \\
&+ G(0,1,0,y) + G(0,y)H(1,0,z)\Big) + 3\Big(- H(0,1,z) - H(0,z)G(1-z,y) \\
&+ G(0,y)H(0,z) + G(1,0,y) + \frac{\pi^2 G(0,y)}{1-y-z}\Big) + \frac{30(1-y-z)G(0,y)}{z^2} + \frac{337}{3} \\
&- \frac{36\zeta_3}{1-y-z} - \frac{\pi^2}{z} - 9\pi^2\Big] \\
&\qquad + \frac{1}{36}\Big(\frac{75y}{z} + \frac{30y^2}{z^2}\Big)\Big[- G(-z,1-z,y) - G(1,0,y) + H(1,z)G(-z,y) \\
&+ H(0,1,z) - H(0,z)G(0,y)\Big] \\
&\qquad + \frac{1}{36}\Big(\frac{75z}{y} + \frac{30z^2}{y^2}\Big)\Big[- G(-z,1-z,y) + H(1,z)G(-z,y) \\
&+ H(0,z)G(1-z,y)\Big] + \frac{5}{6}\Big(\frac{y}{z} + \frac{z}{y}\Big)\Big[G(1-z,y) - H(1,z)\Big] \\
&+ \frac{z}{6(1-y-z)^2}\Big[2G(1,0,y) - H(0,z)G(0,y) - 11z\Big(- H(1,0,z) + G(1,0,y) \\
&- G(0,y)H(0,z)\Big)\Big] - \frac{5}{6}\Big(1 - \frac{1}{1-z}\Big)\Big(\frac{1}{y} + \frac{1}{1-y-z}\Big)H(0,z) \\
&\qquad + \frac{1}{36(1-y-z)^3}\Big(42z(1-y-z) + 24z - 72z(1-y-z)^2\Big)\Big[- H(1,0,z) \\
&+ G(1,0,y) - G(0,y)H(0,z)\Big]
\end{aligned}
$$

$$+\frac{z^2}{6(1-y-z)^4}\Big(-15+26(1-y-z)-16(1-y-z)^2+30z$$
$$-30z(1-y-z)-15z^2\Big)\Big[-H(1,0,z)+G(1,0,y)-G(0,y)H(0,z)$$
$$-\frac{(1-y-z)G(0,y)}{z}-\frac{\pi^2}{6}\Big]$$
$$-\frac{1}{36}\Big(\frac{12}{1-y}(-5-\frac{4}{1-y})+\frac{1}{1-y-z}(\frac{1}{1-y-z}(63-\frac{157z}{2}+\frac{79z^2}{2})$$
$$-12-\frac{11z}{2})+\frac{30y}{z}\Big)G(0,y)$$
$$-\frac{1}{36}\Big(\frac{12}{1-z}(-\frac{4}{1-z}-5)+\frac{30z}{y}+\frac{z}{1-y-z}(\frac{47}{2(1-y-z)}+\frac{181}{2}$$
$$-\frac{90z}{(1-y-z)^2}+\frac{259z}{2(1-y-z)}+\frac{90z^2}{(1-y-z)^2})\Big)H(0,z)$$
$$-\frac{1}{36}\Big(-\frac{48}{1-y}-\frac{48}{1-z}+\frac{33}{1-y-z}$$
$$+\frac{\pi^2 z}{1-y-z}(\frac{4}{(1-y-z)^2}-\frac{2}{1-y-z}-1)\Big)$$
$$+i\pi\Big[\frac{55}{24}\Big(-H(0,z)G(1-z,y)-H(0,z)G(0,y)-H(0,1,z)$$
$$-2H(1,z)G(-z,y)+H(1,z)G(0,y)-H(1,0,z)-G(1-z,0,y)$$
$$+2G(-z,1-z,y)-G(0,1-z,y)+2G(1,0,y)+\frac{11}{6}(-H(0,z)+H(1,z)$$
$$-G(1-z,y)-G(0,y))\Big)+\frac{55yz}{72(1-y-z)^2}-\frac{77\pi^2}{288}+\frac{3\zeta_3}{4}+\frac{185}{24}\Big]\,, \tag{B.7}$$
$$B_\beta^{(2)}=0\,, \tag{B.8}$$
$$C_\beta^{(2)}=0\,, \tag{B.9}$$
$$D_\beta^{(2)}=\frac{y}{12z}\Big(2-\frac{y}{z}\Big)\Big(G(1-z,0,y)-G(-z,1-z,y)+H(0,1,z)+G(0,1-z,y)$$
$$+H(1,z)G(-z,y)-G(0,y)H(1,z)-G(1,0,y)\Big)$$
$$+\frac{z}{12y}\Big(2-\frac{z}{y}\Big)\Big(-G(-z,1-z,y)+H(1,z)G(-z,y)+H(0,z)G(1-z,y)\Big)$$
$$+\frac{1}{12}\Big(\frac{y}{z}+\frac{z}{y}\Big)\Big(-G(1-z,y)+H(1,z)\Big)$$
$$+\frac{5}{24}\Big(-H(1,z)+G(1-z,y)\Big)$$
$$+\frac{z}{12(1-y-z)^4}\Big(4(1-y-z)-2(1-y-z)^2-2(1-y-z)^3+3z$$
$$-10z(1-y-z)+z(1-y-z)^2-6z^2+6z^2(1-y-z)+3z^3\Big)\times$$
$$\Big(-H(1,0,z)+G(1,0,y)-G(0,y)H(0,z)-\frac{G(0,y)(1-y-z)}{z}\Big)$$
$$-\frac{1}{36}\Big(\frac{27}{2}-\frac{9}{(1-y)^2}-\frac{3(1+y)}{z}+\frac{1}{1-y-z}(\frac{3}{z}-\frac{9}{2(1-y-z)}-3$$

$$+\frac{9z}{1-y-z}-\frac{9z^2}{2(1-y-z)}))\Big)G(0,y)$$

$$-\frac{1}{36}\Big(9+\frac{1}{1-z}(\frac{3}{y}-\frac{9}{1-z}+\frac{3}{1-y-z})-\frac{3(1+z)}{y}+\frac{1}{1-y-z}(-3$$

$$+\frac{12z}{1-y-z}-3z+\frac{9z^2}{(1-y-z)^2}-\frac{27z^2}{2(1-y-z)}-\frac{9z^3}{(1-y-z)^2}))\Big)H(0,z)$$

$$-\frac{1}{36}\Big(-\frac{171}{4}-9(\frac{1}{1-y}+\frac{1}{1-z})+\frac{21}{2(1-y-z)}+36\zeta_3+\frac{9yz}{(1-y-z)^2}$$

$$+\frac{\pi^2 z}{(1-y-z)^4}(2(1-y-z)-(1-y-z)^2-(1-y-z)^3+\frac{3z}{2}$$

$$-5z(1-y-z)+\frac{z(1-y-z)^2}{2}-3z^2+3z^2(1-y-z)+\frac{3z^3}{2})\Big)$$

$$+i\frac{\pi}{4}\ , \tag{B.10}$$

$$E_\beta^{(2)}=\frac{1}{6}\Big[\frac{1}{2}\Big(G(0,1-z,y)H(0,z)-H(0,1,z)G(0,y)+H(0,z)G(1-z,0,y)\Big)$$

$$+G(1-z,1,0,y)+G(1,1-z,0,y)+G(1,0,1-z,y)+H(1,0,z)G(1-z,y)$$

$$-G(1,0,y)H(1,z)-H(0,1,z)G(1-z,y)$$

$$+2\Big(+G(-z,-z,1-z,y)-G(1-z,1-z,0,y)-G(1-z,0,1-z,y)$$

$$-G(1-z,0,0,y)-H(1,1,0,z)+H(1,z)G(1-z,0,y)+H(1,0,1,z)$$

$$-H(1,0,0,z)-H(1,z)G(-z,-z,y)-G(0,1-z,1-z,y)-G(0,1-z,0,y)$$

$$+G(0,1-z,y)H(1,z)-H(0,1,z)G(-z,y)+H(0,1,1,z)-G(0,0,1-z,y)$$

$$-H(0,0,z)G(1-z,y)-H(0,0,z)G(0,y)+G(0,0,y)H(1,z)-G(0,0,y)H(0,z)$$

$$-H(0,z)G(1-z,1-z,y)-G(0,y)H(1,1,z)\Big)$$

$$+3\Big(G(1-z,-z,1-z,y)-H(1,z)G(1-z,-z,y)-H(0,1,0,z)\Big)$$

$$+4\Big(G(-z,1-z,1-z,y)+H(1,1,z)G(-z,y)+G(1,0,0,y)$$

$$-H(1,z)G(-z,1-z,y)+G(0,1,0,y)\Big)\Big]$$

$$+\frac{1}{36}\Big[-20\Big(G(-z,1-z,y)-H(1,z)G(-z,y)+G(1,0,y)\Big)$$

$$-21\Big(-H(0,1,z)-H(1,0,z)\Big)+33\Big(-G(1-z,1-z,y)-H(1,1,z)$$

$$+H(1,z)G(1-z,y)-H(0,0,z)-G(0,0,y)\Big)-G(1-z,0,y)-G(0,1-z,y)$$

$$-H(0,z)G(1-z,y)+G(0,y)H(1,z)-G(0,y)H(0,z)+\frac{35}{6}\Big(G(1-z,y)$$

$$-H(1,z)\Big)-9G(0,y)H(1,0,z)+\frac{238}{3}H(0,z)+\frac{575}{6}G(0,y)\Big]$$

$$+\frac{\pi^2}{36}\Big(-\frac{1}{4}G(1-z,y)-\frac{7}{4}H(1,z)-\frac{5}{4}H(0,z)-\frac{9}{4}G(0,y)+\frac{1879}{24}\Big)-\frac{13\zeta_3}{24}$$

$$
\begin{aligned}
&+\frac{yz}{12(1-y-z)^2}\Big[G(1-z,0,y)+G(0,1-z,y)+H(0,1,z)\\
&+H(0,z)G(1-z,y)-G(0,y)H(1,z)+2\Big(-G(-z,1-z,y)+H(1,z)G(-z,y)\\
&-\frac{\pi^2}{1-y-z}H(1,z)\Big)+\frac{6}{1-y-z}\Big(-G(0,1,0,y)-G(0,y)H(1,0,z)\Big)\\
&+\frac{39}{6}\Big(G(1-z,y)-H(1,z)\Big)-\frac{12H(1,1,0,z)}{1-y-z}-\frac{\pi^2G(0,y)}{1-y-z}+\frac{12\zeta_3}{1-y-z}\\
&+\frac{11\pi^2z^2}{(1-y-z)^2}-\frac{374}{9}+3\pi^2\Big]\\
&+\frac{z}{36(1-y-z)^3}\Big(3(1-y-z)^2+3z(1-y-z)+2z^2\Big)\Big[6\Big(G(1,1,0,y)\\
&-H(1,1,0,z)\Big)+3\Big(-H(0,1,0,z)-G(0,1,0,y)-H(0,z)G(1,0,y)\\
&-G(0,y)H(1,0,z)\Big)+\pi^2\Big(-G(1,y)-H(1,z)+\frac{H(0,z)}{2}-\frac{G(0,y)}{2}\Big)\Big]\\
&+\frac{1}{36(1-y-z)^3}\Big(3(1-y-z)^2-3(1-y-z)+2\Big)\Big[6\Big(H(1,1,0,z)-\zeta_3\Big)\\
&+3\Big(G(0,1,0,y)+G(0,y)H(1,0,z)\Big)+\pi^2H(1,z)+\frac{\pi^2G(0,y)}{2}\Big]\\
&+\frac{y}{36z}\Big(42+\frac{33y}{z}\Big)\Big[-G(1-z,0,y)+G(-z,1-z,y)-H(1,z)G(-z,y)\\
&-G(0,1-z,y)-H(0,1,z)+G(0,y)H(1,z)+G(1,0,y)\Big]\\
&+\frac{z}{36y}\Big(42+\frac{33z}{y}\Big)\Big[+G(-z,1-z,y)-H(1,z)G(-z,y)-H(0,z)G(1-z,y)\Big]\\
&+\frac{11}{12}\Big(\frac{y}{z}+\frac{z}{y}\Big)\Big[-G(1-z,y)+H(1,z)\Big]\\
&+\frac{z^2}{36(1-y-z)^2}\Big(-\frac{12}{z(1-y-z)}-\frac{15}{z}+\frac{39(1-y-z)}{z}+\frac{99}{(1-y-z)^2}\\
&-\frac{186}{1-y-z}+141-\frac{198z}{(1-y-z)^2}+\frac{198z}{1-y-z}+\frac{99z^2}{(1-y-z)^2}\Big)\Big[-H(1,0,z)\\
&+G(1,0,y)-G(0,y)H(0,z)-\frac{1-y-z}{z}G(0,y)\Big]\\
&-\frac{1}{36(1-y-z)^2}\Big(-6+6(1-y-z)-3z^2\Big)H(1,0,z)\\
&+\frac{1}{36(1-y-z)^2}\Big(6-6(1-y-z)-27z+3z(1-y-z)\Big)G(1,0,y)\\
&-\frac{z}{36(1-y-z)^2}\Big(-12-3z\Big)G(0,y)H(0,z)\\
&-\frac{1}{36}\Big(\frac{1}{(1-y)^2}(39+6(1-y))+\frac{33}{z}(\frac{1}{1-y-z}-1)+\frac{1}{(1-y-z)^2}(-\frac{87}{2}\\
&+15(1-y-z)+\frac{173z}{2}-\frac{125z(1-y-z)}{2}-43z^2)-\frac{33y}{z}\Big)G(0,y)
\end{aligned}
$$

$$- \frac{1}{36}\Big(\frac{33}{y}(\frac{1}{1-z} - 1 - z) + \frac{39}{(1-z)^2} + \frac{33}{(1-z)(1-y-z)} + \frac{6}{1-z}$$

$$- \frac{1}{(1-y-z)^2}(33(1-y-z) + \frac{25z}{2} + \frac{131z(1-y-z)}{2} - \frac{99z^2}{1-y-z} + 142z^2$$

$$+ \frac{99z^3}{1-y-z})\Big)H(0,z)$$

$$- \frac{1}{36}\Big(\frac{1115}{12} + 39(\frac{1}{1-y} + \frac{1}{1-z}) + \frac{\pi^2}{(1-y-z)^2}(- 1 - \frac{45(1-y-z)}{2\pi^2}$$

$$+ (1-y-z) - \frac{2z}{(1-y-z)} + \frac{13z}{2} - \frac{9z(1-y-z)}{2} + \frac{33z^2}{2(1-y-z)^2} - \frac{31z^2}{1-y-z}$$

$$+ 12z^2 - \frac{33z^4}{2(1-y-z)^2})\Big)$$

$$+ i\pi\Big[\frac{5}{12}\Big(+ H(0,z)G(1-z,y) + H(0,z)G(0,y) + H(0,1,z)$$

$$+ 2H(1,z)G(-z,y) - H(1,z)G(0,y) + H(1,0,z) + G(1-z,0,y)$$

$$- 2G(-z,1-z,y) + G(0,1-z,y) - 2G(1,0,y) + \frac{11}{3}(+ H(0,z) - H(1,z)$$

$$+ G(1-z,y) + G(0,y))\Big) - \frac{65yz}{72(1-y-z)^2} + \frac{7\pi^2}{144} - \frac{71}{18}\Big]\,, \qquad \text{(B.11)}$$

$$F_\beta^{(2)} = \frac{1}{12}\Big(G(1-z,1-z,y) + H(1,1,z) - H(1,z)G(1-z,y) + H(0,0,z) + G(0,0,y)\Big)$$

$$+ \frac{1}{36}\Big(+ G(1-z,0,y) - H(1,0,z) + G(0,1-z,y) - H(0,1,z)$$

$$+ H(0,z)G(1-z,y) - G(0,y)H(1,z) + G(0,y)H(0,z)\Big)$$

$$+ \frac{5}{54}\Big(- G(0,y) - H(0,z) + H(1,z) - G(1-z,y)\Big)$$

$$+ \frac{yz}{36(1-y-z)^2}\Big(- G(0,y) - H(0,z) + 3H(1,z) - 3G(1-z,y) + \frac{10}{3}\Big) - \frac{29\pi^2}{144}$$

$$+ i\frac{5\pi}{36}\Big[- H(0,z) + H(1,z) - G(1-z,y) - G(0,y) + 2 + \frac{yz}{(1-y-z)^2}\Big]\,, \qquad \text{(B.12)}$$

$$A_\gamma^{(2)} = \frac{1}{2}\Big[\frac{G(1-z,0,0,y)}{3} + \frac{1}{2}\Big(- G(1-z,1,1,0,y) + G(1,1-z,-z,1-z,y)$$

$$- G(1,1-z,-z,y)H(1,z) + G(1,1-z,1,0,y) + H(1,1,0,0,z)$$

$$- G(1,0,-z,y)H(1,z) - G(1,0,1,0,y) + H(1,0,1,z)G(1,y) + G(1,0,-z,1-z,y)$$

$$- H(1,0,0,z)G(1-z,y) + G(1,0,y)H(1,0,z) + G(0,-z,1-z,y)H(0,z)$$

$$- G(0,-z,y)H(1,0,z) + G(0,1-z,-z,1-z,y) - G(0,1-z,-z,y)H(1,z)$$

$$+ G(0,1-z,0,y)H(0,z) + G(0,1-z,y)H(1,0,z) + G(0,1,0,y)H(0,z)$$

$$- H(0,1,z)G(1,1-z,y) - H(0,1,z)G(0,-z,y) + H(0,1,z)G(0,1-z,y)$$

$$+ H(0,0,1,z)G(1,y) + H(0,0,1,z)G(0,y) - H(0,0,z)G(1-z,0,y)$$

$$+ G(0,0,y)H(1,0,z) + G(0,0,y)H(0,0,z) + H(0,z)G(1-z,1,0,y)$$

$$- H(0,z)G(1-z,0,0,y) - H(0,z)G(1,1-z,0,y) - H(0,z)G(1,0,1-z,y)$$

$$
\begin{aligned}
&+G(0,y)H(1,0,1,z)+G(0,y)H(1,0,0,z)\Big)\\
&\qquad+\frac{3}{4}\Big(-G(1,1-z,0,y)-G(1,0,1-z,y)+G(1,0,y)H(1,z)\\
&-H(0,0,z)G(0,y)\Big)\\
&\qquad+G(1-z,-z,1-z,0,y)+G(1-z,-z,0,1-z,y)-G(1-z,1-z,0,0,y)\\
&-G(1-z,1,1-z,0,y)-G(1-z,1,0,1-z,y)+G(1-z,0,-z,1-z,y)\\
&-G(1-z,0,1-z,0,y)+G(1-z,0,1,0,y)-G(1-z,0,0,1-z,y)\\
&+G(1,1-z,0,0,y)+G(1,1,1,0,y)-G(1,1,0,0,y)+G(1,0,1-z,0,y)\\
&+H(1,0,1,z)G(1-z,y)+H(1,0,0,z)G(-z,y)+G(1,0,0,1-z,y)\\
&-H(1,0,0,z)G(-z,y)-G(1,0,0,y)H(1,z)+H(1,0,z)G(-z,0,y)\\
&-H(1,0,z)G(1-z,-z,y)-H(1,z)G(1-z,-z,0,y)+H(1,z)G(1-z,1,0,y)\\
&-H(1,z)G(1-z,0,-z,y)+H(1,z)G(1-z,0,0,y)-G(0,1-z,1-z,0,y)\\
&-G(0,1-z,1-z,y)H(0,z)+G(0,1-z,1,0,y)-G(0,1-z,0,1-z,y)\\
&+G(0,1-z,0,y)H(1,z)+H(0,1,1,0,z)-G(0,1,1,0,y)+H(0,1,1,z)G(0,y)\\
&-H(0,1,0,z)G(1-z,y)+H(0,1,z)G(-z,0,y)-H(0,1,z)G(1-z,0,y)\\
&-G(0,0,1-z,1-z,y)+G(0,0,1-z,y)H(1,z)-H(0,0,1,1,z)\\
&-H(0,0,z)G(1-z,1-z,y)-G(0,0,y)H(1,1,z)-H(0,z)G(-z,1-z,0,y)\\
&-H(0,z)G(-z,0,1-z,y)+H(0,z)G(1-z,-z,1-z,y)\\
&+H(0,z)G(1-z,1-z,0,y)+G(0,y)H(1,1,0,z)\\
&\qquad+\frac{3}{2}\Big(G(1-z,1,0,0,y)-H(1,1,0,1,z)-H(1,1,0,z)+H(1,0,1,0,z)\\
&-H(1,0,0,1,z)-H(1,0,z)G(1-z,0,y)-G(0,-z,1-z,0,y)\\
&-G(0,-z,0,1-z,y)+G(0,-z,0,y)H(1,z)+H(0,0,1,0,z)\Big)\\
&\qquad+2\Big(G(-z,1-z,1-z,0,y)+G(-z,1-z,0,1-z,y)\\
&+G(-z,0,1-z,1-z,y)+G(1-z,1-z,-z,1-z,y)-G(1-z,1-z,1,0,y)\\
&+H(1,1,1,0,z)+H(1,1,0,z)G(-z,y)+H(1,1,z)G(-z,0,y)\\
&-H(1,0,1,z)G(-z,y)-H(1,0,z)G(-z,1-z,y)+H(1,0,z)G(1-z,1-z,y)\\
&-H(1,z)G(-z,1-z,0,y)+H(1,z)G(-z,-z,y)-H(1,z)G(1-z,1-z,-z,y)\\
&-H(1,z)G(-z,0,1-z,y)+G(0,-z,1-z,1-z,y)-G(0,-z,1-z,y)H(1,z)\\
&+G(0,-z,y)H(1,1,z)-H(0,1,1,z)G(-z,y)+H(0,1,z)G(-z,1-z,y)\\
&-H(0,1,z)G(1-z,1-z,y)+H(0,z)G(-z,1-z,1-z,y)\Big)\\
&\qquad+3\Big(H(0,1,z)G(1-z,-z,y)+H(0,0,1,z)G(1-z,y)\Big)\\
&\qquad+4\Big(G(-z,-z,-z,1-z,y)-G(-z,-z,1-z,1-z,y)\\
&-G(-z,1-z,-z,1-z,y)-G(1-z,-z,-z,1-z,y)-G(1-z,1-z,0,y)\\
&-G(1-z,0,1-z,y)-H(1,1,z)G(-z,-z,y)-H(1,z)G(-z,-z,-z,y)
\end{aligned}
$$

$$
\begin{aligned}
&+H(1,z)G(-z,-z,1-z,y)+H(1,z)G(-z,1-z,-z,y)\\
&+H(1,z)G(1-z,-z,-z,y)+H(1,z)G(1-z,0,y)-G(0,1-z,1-z,y)\\
&+G(0,1-z,y)H(1,z)-H(0,0,1,z)G(-z,y)+H(0,1,1,z)\\
&-H(0,1,z)G(-z,-z,y)-H(0,z)G(1-z,1-z,y)-G(0,y)H(1,1,z)\Big)\\
&\qquad+6\Big(-G(1-z,0,0,y)+H(0,0,1,z)\Big)+8\Big(G(-z,1-z,1-z,y)\\
&+H(1,1,z)G(-z,y)-H(1,z)G(-z,1-z,y)\Big)\Big]\\
&\qquad+\frac{1}{36}\Big[30\Big(\frac{+H(0,z)G(1,0,y)}{2}+G(-z,1-z,0,y)+G(-z,0,1-z,y)\\
&-H(1,z)G(-z,0,y)\Big)+39\Big(-G(-z,-z,1-z,y)+G(1-z,1,0,y)\\
&+G(1-z,0,0,y)-H(1,0,0,z)+G(0,1-z,0,y)-H(0,1,0,z)\\
&-H(0,0,1,z)+H(0,0,z)G(1-z,y)-G(0,0,y)H(1,z)\Big)\\
&+63\Big(-H(1,1,0,z)G(1-z,y)+G(1,0,0,y)-G(0,0,1-z,y)\Big)\\
&+33\Big(\frac{1}{2}(H(0,z)G(-z,1-z,y)-H(0,z)G(1-z,0,y)-H(1,0,z)G(-z,y))\\
&+G(1,1,0,y)+H(1,0,z)G(1-z,y)-2H(0,1,z)G(0,y)\\
&+6H(0,1,z)\Big)+3\Big(+\frac{G(0,y)H(1,0,z)}{2}+H(1,z)G(-z,-z,y)\Big)\\
&-\frac{171}{2}\Big(-H(1,0,1,z)+H(0,1,z)G(1-z,y)\Big)+\frac{51H(1,0,0,z)}{2}\\
&+\frac{455}{4}\Big(-G(1-z,1-z,y)-H(1,1,z)+H(1,z)G(1-z,y)\Big)\\
&+102\Big(-G(0,1-z,0,y)+G(0,0,y)H(1,z)\Big)+45\Big(\frac{1}{2}(G(0,1-z,y)H(0,z)\\
&+H(0,1,z)G(-z,y))-H(0,1,0,1,z)-H(0,0,0,1,z)\Big)+\frac{21H(0,1,0,z)}{2}\\
&+48G(0,1,0,y)+\frac{87}{2}\Big(G(0,-z,1-z,y)-G(0,-z,y)H(1,z)\Big)\\
&-\frac{315}{2}\Big(H(1,z)G(1-z,-z,y)-G(1-z,-z,1-z,y)-G(0,y)\Big)\\
&+133\Big(G(0,1-z,y)+G(1-z,0,y)-G(0,y)H(1,z)\Big)\\
&+\frac{121H(1,0,z)}{4}-\frac{151H(0,0,z)}{2}-\frac{177H(0,0,z)G(1-z,y)}{2}-80G(0,0,y)\\
&-\frac{635G(1,0,y)}{4}-\frac{127G(0,y)H(0,z)}{4}+115H(0,z)G(1-z,y)+\frac{459H(0,z)}{4}\\
&-313\Big(-H(1,z)G(-z,y)+G(-z,1-z,y)\Big)\Big]\\
&\qquad+\frac{\pi^2}{36}\Big[18\Big(G(1-z,0,y)-G(1,0,y)+H(0,1,z)\Big)+G(1-z,y)\\
&+\frac{3G(1-z,1,y)}{2}-\frac{21}{2}\Big(-H(1,1,z)-H(0,1,z)\Big)+3\Big(-G(1,1,y)+G(0,1,y)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-6H(0,z)G(1-z,y)+42G(0,y)H(0,z)-18G(0,y)H(1,z)\\
&-\frac{27}{2}\Big(H(1,z)G(1-z,y)+2G(-z,1-z,y)-2H(1,z)G(-z,y)\Big)-\frac{11G(1,y)}{2}\\
&+12G(1-z,1-z,y)+66\Big(-G(0,0,y)-H(0,0,z)\Big)+39\Big(-H(0,z)G(0,y)\\
&-G(1-z,y)+H(1,z)+\frac{G(0,1-z,y)}{2}\Big)-\frac{135}{2}\Big(-H(0,0,z)-G(0,0,y)\Big)\\
&+\frac{9}{2}\Big(H(1,0,z)-G(1,0,y)\Big)-\frac{3}{8}\Big(G(0,y)+H(0,z)\Big)\Big]\\
&\qquad+\frac{z}{8y}\Big(\frac{z}{y}-2\Big)\Big[-2G(-z,-z,1-z,y)+G(1-z,-z,1-z,y)\\
&-H(1,0,z)G(-z,y)+2H(1,z)G(-z,-z,y)-H(1,z)G(1-z,-z,y)\\
&-G(0,-z,1-z,y)+G(0,-z,y)H(1,z)+G(0,1-z,y)H(0,z)\\
&+H(0,1,z)G(-z,y)-H(0,1,z)G(1-z,y)+H(0,z)G(-z,1-z,y)\\
&-H(0,0,z)G(1-z,y)\Big]\\
&\qquad+\frac{1}{8y}\Big(1-z\Big)\Big[-H(1,0,z)-3H(0,0,z)-2G(0,0,y)-\frac{16G(0,y)}{3}\\
&+\frac{10zH(0,z)}{y}-\frac{37}{3}\Big(H(0,z)G(1-z,y)+H(1,z)G(-z,y)-G(-z,1-z,y)\Big)\\
&+\frac{55H(0,z)}{6}+2\Big(G(0,0,y)+H(0,0,z)\Big)-2\pi^2\Big]\\
&\qquad-\frac{5(1-y)G(0,y)}{8z}+\frac{1}{8y^2}\Big(2z-1-2y\Big)\Big[-G(1-z,-z,1-z,y)\\
&+H(1,z)G(1-z,-z,y)+G(0,-z,1-z,y)-G(0,-z,y)H(1,z)\\
&-G(0,1-z,y)H(0,z)+H(0,1,z)G(1-z,y)+H(0,0,z)G(1-z,y)\Big]\\
&\qquad+\frac{1}{36}\Big(\frac{9}{2y}(\frac{1}{z}-2+z)+\frac{9}{z^2}(-\frac{5}{2}-4z+5y+\frac{7yz}{2}-\frac{5y^2}{2})-24\pi^2\Big)\times\\
&\Big[G(1-z,0,y)-G(1,0,y)+G(0,1-z,y)+H(0,1,z)-G(0,y)H(1,z)\\
&+H(1,z)G(-z,y)-G(-z,1-z,y)\Big]\\
&\qquad+\frac{1}{8y}\Big[G(1,0,y)-H(0,1,z)-G(0,y)H(0,z)+\frac{2}{9}\Big(39H(0,z)G(1-z,y)\\
&-\frac{237H(0,z)}{4}+\frac{69H(1,z)G(-z,y)}{2}-\frac{69G(-z,1-z,y)}{2}\Big)\Big]\\
&\qquad+\frac{1}{36}\Big(\frac{9z}{y^3}(5-10z+5z^2)+\frac{6}{y^2}(-4-6z+10z^2)\Big)\Big[H(0,z)G(1-z,y)\\
&+H(1,z)G(-z,y)-\frac{H(1,z)y}{z}+\frac{G(1-z,y)y}{z}-G(-z,1-z,y)\Big]\\
&\qquad+\frac{1}{36}\Big(\frac{9965}{24}+\frac{3}{2yz}(1-z^2)+\frac{45}{z}-\frac{48}{(y+z)^2}-\frac{60}{y+z}+72\zeta_3+\frac{85\pi^2}{2}-\frac{45y}{2z}\Big)\times
\end{aligned}
$$

$$
\begin{aligned}
&\Big[G(1-z,y)-H(1,z)\Big]\\
&\quad+\frac{\pi^2}{48y^2}\Big(1-2z+z^2+2y-2yz\Big)G(1-z,y)+\frac{19H(0,z)}{48(1-z)}\\
&\quad-\frac{1}{36}\Big(\frac{260437}{288}+\frac{1}{y}(-\frac{269}{2}-9\pi^2+\frac{269z}{2}+\frac{39z\pi^2}{4})+\frac{48}{y+z}-\frac{99\zeta_4}{2}-\frac{319\zeta_3}{2}\\
&-\frac{439\pi^2}{2}-\frac{\pi^4}{8}\Big)\\
&\quad+i\pi\Big[\frac{55}{24}\Big(+H(0,z)G(1-z,y)+H(0,1,z)+2H(1,z)G(-z,y)\\
&-H(1,z)G(0,y)+G(1-z,0,y)-2G(-z,1-z,y)+G(0,1-z,y)-G(1,0,y)\Big)\\
&+\frac{715}{144}\Big(-H(1,z)+G(1-z,y)\Big)+\frac{275}{72}\Big(H(0,z)+G(0,y)\Big)+\frac{11\pi^2}{144}-2\zeta_3\\
&-\frac{16499}{864}+\frac{55(1-z)}{48y}\Big], \qquad\qquad (B.13)
\end{aligned}
$$

$$
\begin{aligned}
B_\gamma^{(2)} &= \frac{1}{2}\Big[\frac{1}{2}\Big(-G(1,1-z,-z,1-z,y)+G(1,1-z,-z,y)H(1,z)+H(1,1,0,1,z)\\
&+H(1,1,0,z)G(1,y)+G(1,0,-z,y)H(1,z)-G(1,0,-z,1-z,y)\\
&-H(1,0,1,z)G(1,y)-H(1,0,0,1,z)-H(1,0,z)G(1,1-z,y)\\
&-G(1,0,y)H(1,0,z)+G(0,-z,1-z,0,y)-G(0,-z,1-z,y)H(0,z)\\
&+G(0,-z,0,1-z,y)-G(0,-z,0,y)H(1,z)+G(0,-z,y)H(1,0,z)\\
&+G(0,1-z,-z,1-z,y)-G(0,1-z,-z,y)H(1,z)-G(0,1-z,1,0,y)\\
&-G(0,1-z,0,y)H(0,z)-G(0,1,1-z,0,y)-G(0,1,0,1-z,y)\\
&+H(0,1,0,z)G(1,y)+G(0,1,0,y)H(1,z)-G(0,1,0,y)H(0,z)\\
&+H(0,1,z)G(1,1-z,y)-H(0,1,z)G(1,0,y)+H(0,1,z)G(0,-z,y)\\
&-H(0,1,z)G(0,1-z,y)+H(0,0,1,0,z)-H(0,0,1,z)G(1,y)\\
&+H(0,z)G(1,1-z,0,y)+G(0,y)H(1,1,0,z)\Big)\\
&\quad+\frac{3}{4}\Big(G(1-z,1,0,y)-G(1,1-z,0,y)-G(1,0,1-z,y)+G(1,0,y)H(1,z)\\
&-H(0,1,z)G(0,y)-H(0,z)G(1-z,0,y)+G(1-z,-z,1-z,y)\\
&-H(1,0,z)G(-z,y)-H(1,z)G(1-z,-z,y)+H(0,1,z)G(-z,y)+H(0,0,1,z)\\
&+H(0,z)G(-z,1-z,y)\Big)\\
&\quad+H(0,1,1,0,z)+H(0,1,0,z)G(0,y)+G(0,0,-z,1-z,y)\\
&-G(0,0,-z,y)H(1,z)-G(0,0,1-z,y)H(0,z)-G(0,0,1,0,y)\\
&+H(0,z)G(1,0,1-z,y)+G(0,y)H(1,0,1,z)\\
&\quad+\frac{3}{2}\Big(H(0,0,1,z)G(0,y)-H(0,0,0,1,z)-G(-z,-z,1-z,y)\\
&+H(1,z)G(-z,-z,y)-G(0,1-z,y)H(0,z)+H(0,0,z)G(1-z,y)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 2\Big(H(1,1,1,0,z) - G(1,1,1,0,y) + H(1,1,0,0,z) + G(1,1,0,0,y) \\
&+ H(1,0,1,0,z) + G(1,0,1,0,y) + H(1,0,0,z) + G(0,1,1,0,y) \\
&+ H(0,0,z)G(0,y) + G(0,0,y)H(0,0,z) - H(0,z)G(1,0,0,y) \\
&+ G(0,y)H(1,0,0,z)\Big) \\
&\quad + 3\Big(H(1,z)G(-z,y) - G(-z,1-z,y)\Big)\Big] \\
&\quad + \frac{1}{36}\Big[-33G(1,1,0,y) + 63\Big(-G(1,0,0,y) + G(0,0,y)H(0,z)\Big) \\
&+ 27H(1,0,z)G(1-z,y) - 48G(0,1,0,y) + 12H(0,z)G(1,0,y) \\
&+ \frac{57G(0,y)H(1,0,z)}{2} + \frac{81}{2}\Big(G(0,-z,1-z,y) - G(0,-z,y)H(1,z)\Big) \\
&- \frac{39H(1,1,0,z)}{2} - \frac{27H(0,1,z)G(1-z,y)}{2} + \frac{69H(0,1,0,z)}{2} \\
&+ \frac{455}{4}\Big(-G(0,y)H(0,z) + G(1,0,y) - H(1,0,z)\Big) + \frac{399G(0,y)}{4} \\
&+ \frac{135H(0,z)G(1-z,y)}{4} + \frac{201H(0,z)}{4} + \frac{2491}{12}\Big(-H(1,z) + G(1-z,y)\Big)\Big] \\
&\quad + \frac{\pi^2}{36}\Big[\frac{3}{2}\Big(-G(1,1-z,y) + G(1,y)H(1,z) + H(0,z)G(1,y) - G(0,y)H(1,z) \\
&+ G(0,1-z,y)\Big) + 6\Big(G(1,1,y) - G(0,1,y)\Big) + \frac{9H(1,1,z)}{2} + \frac{H(0,z)}{4} + \frac{11G(1,y)}{2} \\
&+ 3G(0,y)H(0,z) - \frac{9G(1,0,y)}{2} + \frac{9H(1,0,z)}{2} + 7G(0,y) + \frac{9H(0,1,z)}{2} - \frac{5H(1,z)}{8} \\
&+ \frac{33G(1-z,y)}{8}\Big] \\
&\quad + \frac{\zeta_3}{36}\Big[27\Big(-G(1,y) + G(0,y) + H(0,z)\Big) + 36H(1,z) - 9G(1-z,y)\Big] \\
&\quad + \frac{z}{4y}\Big(4 + \frac{z}{y}\Big)\Big[-G(-z,-z,1-z,y) - \frac{H(1,0,z)G(-z,y)}{2} \\
&+ H(1,z)G(-z,-z,y) + \frac{H(0,1,z)G(-z,y)}{2} + \frac{+H(0,z)G(-z,1-z,y)}{2}\Big] \\
&\quad + \frac{z}{8(1-y)}\Big(4 - \frac{z}{1-y}\Big)\Big[-G(1-z,-z,1-z,y) + H(1,1,0,z) \\
&- H(1,0,1,z) - H(1,0,z)G(1-z,y) + H(1,z)G(1-z,-z,y) - G(0,-z,1-z,y) \\
&+ G(0,-z,y)H(1,z) + G(0,1-z,y)H(0,z) + H(0,1,0,z) + H(0,1,z)G(1-z,y) \\
&- H(0,0,1,z) + \frac{\pi^2}{6}\Big(-G(1-z,y) + H(1,z) + H(0,z)\Big)\Big] \\
&\quad + \frac{(z-1)}{4y}\Big[+3\Big(H(1,0,z)G(1-z,y) + G(0,-z,1-z,y) - G(0,-z,y)H(1,z) \\
&- G(0,1-z,y)H(0,z) + H(0,0,z)G(1-z,y)\Big) + \frac{8G(0,y)}{3} + \frac{G(0,y)H(0,z)}{2}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{G(1,0,y)}{2}+\frac{H(1,0,z)}{2}-\frac{H(0,z)z}{y}+\frac{\pi^2 G(1-z,y)}{2}+\frac{\pi^2}{2}\Big] \\
& \quad +\frac{1}{8}\Big(\frac{z}{y}+\frac{z}{1-y}\Big)\Big[-H(0,1,z)+H(1,0,z)\Big] \\
& -\frac{3z}{8(1-y-z)}\Big[-G(0,y)H(0,z)+G(1,0,y)-H(1,0,z)\Big]-\frac{(1-y)}{16z}G(0,y) \\
& \quad +\frac{1}{36}\Big(\frac{81}{4}+\frac{9}{2}(\frac{1}{y}-\frac{z}{y}-\frac{1}{2z^2})+\frac{27}{2z}+\frac{9y}{2z^2}-\frac{27y}{2z}-\frac{9y^2}{4z^2}\Big)\Big[G(1-z,0,y) \\
& -G(1,0,y)+G(0,1-z,y)+H(0,1,z)-G(0,y)H(1,z)\Big] \\
& \quad +\frac{z(1+z)}{8(1-y)}\Big[G(-z,1-z,y)-H(1,z)G(-z,y)\Big] \\
& \quad +\frac{1}{36}\Big(\frac{9}{2y^2}(1-2z+z^2-\frac{y}{2z})-\frac{63}{4y}+\frac{18z}{y}+\frac{9}{2z}-\frac{9y}{4z}\Big)\Big[G(1-z,y)-H(1,z)\Big] \\
& \quad +\frac{1}{36}\Big(\frac{9z}{2y^3}(-1+2z-z^2)+\frac{1}{y^2}(\frac{63}{4}-\frac{9z}{2}-\frac{45z^2}{4})+\frac{1}{y}(-\frac{45}{2}+\frac{9z^2}{2})+\frac{9}{4z^2} \\
& -\frac{27}{2z}-\frac{9y}{2z^2}+\frac{27y}{2z}+\frac{9y^2}{4z^2}\Big)\Big[G(-z,1-z,y)-H(1,z)G(-z,y)\Big] \\
& \quad +\frac{1}{36}\Big(\frac{9z}{2y^3}(1-2z+z^2)+\frac{1}{y^2}(-\frac{63}{4}+\frac{9z}{2}+\frac{45z^2}{4})+\frac{1}{y}(18+9z-\frac{9z^2}{2}) \\
& -\frac{9z^2}{2(1-y)}\Big)H(0,z)G(1-z,y) \\
& \quad -\frac{1}{36}\Big(\frac{9z(1-z)}{2y^2}+\frac{1}{y}(\frac{57}{4}+\frac{3z}{4})-\frac{24}{1-z}\Big)H(0,z) \\
& \quad -\frac{1}{36}\Big(\frac{48443}{72}+\frac{1}{y}(139+\frac{15\pi^2}{4})(z-1)-\frac{27z^2\zeta_3}{2(1-y)^2}+\frac{54z\zeta_3}{1-y}-\frac{3z\pi^2}{4(1-y)} \\
& +\frac{135\zeta_4}{4}-\frac{761\zeta_3}{4}+\frac{77\pi^2}{6}+\frac{\pi^4}{16}-\frac{9z\pi^2}{4(1-y-z)}\Big) \\
& \quad +i\pi\Big[\frac{55}{24}\Big(-H(0,z)G(0,y)-H(1,0,z)+G(1,0,y)+\frac{1-z}{2y}\Big)-\frac{4291}{432} \\
& +\frac{\zeta_3}{4}-\frac{107\pi^2}{288}\Big]\,, \qquad \text{(B.14)}
\end{aligned}
$$

$$
\begin{aligned}
C_\gamma^{(2)} = & \frac{1}{2}\Big[\frac{1}{2}\Big(+G(1-z,1,1,0,y)+G(1-z,1,0,0,y)-G(1,1-z,1,0,y) \\
& +H(1,1,0,0,z)+H(1,1,0,z)G(1,y)-H(1,1,0,z)G(1-z,y)-H(1,0,1,0,z) \\
& -H(1,0,0,z)G(1-z,y)-H(1,0,z)G(1-z,0,y)-H(1,0,z)G(1,1-z,y) \\
& -G(0,1-z,1,0,y)-G(0,1-z,y)H(1,0,z)+G(0,1,1-z,0,y) \\
& +G(0,1,0,1-z,y)+H(0,1,0,z)G(1,y)-G(0,1,0,y)H(1,z) \\
& +H(0,1,z)G(1,0,y)-H(0,0,z)G(1-z,0,y)+G(0,0,y)H(1,0,z) \\
& +G(0,0,y)H(0,0,z)+H(0,z)G(1-z,1,0,y)-H(0,z)G(1-z,0,0,y) \\
& +H(0,z)G(1,0,1-z,y)+G(0,y)H(1,1,0,z)+G(0,y)H(1,0,0,z)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{4}\Big(-G(1-z,1,0,y)-H(1,0,0,z)-H(0,0,z)G(0,y)\\
&+H(0,z)G(1-z,0,y)-G(0,y)H(1,0,z)\Big)\\
&-G(1-z,1,1-z,0,y)-G(1-z,1,0,1-z,y)-G(1,1-z,-z,1-z,y)\\
&+G(1,1-z,-z,y)H(1,z)+G(1,1-z,0,0,y)+G(1,1,1,0,y)-G(1,1,0,0,y)\\
&-G(1,0,-z,1-z,y)+G(1,0,-z,y)H(1,z)+G(1,0,1-z,0,y)\\
&-H(1,0,1,z)G(1-z,y)-H(1,0,1,z)G(1,y)+G(1,0,0,1-z,y)\\
&-G(1,0,0,y)H(1,z)+H(1,z)G(1-z,1,0,y)+G(0,1-z,-z,1-z,y)\\
&-G(0,1-z,-z,y)H(1,z)-H(0,1,1,0,z)-G(0,1,1,0,y)\\
&-H(0,1,z)G(1-z,0,y)+H(0,1,z)G(1,1-z,y)-H(0,1,z)G(0,1-z,y)\\
&-G(0,0,-z,1-z,y)+G(0,0,-z,y)-H(0,0,1,0,z)+G(0,0,1,0,y)\\
&-G(0,0,-z,y)+G(0,0,-z,y)H(1,z)-H(0,0,1,z)G(1,y)+H(0,0,0,1,z)\\
&-H(0,0,z)G(0,1-z,y)+G(0,0,y)H(0,1,z)+H(0,z)G(1-z,0,1-z,y)\\
&+\frac{3}{2}\Big(-G(1,0,1,0,y)+H(0,1,0,1,z)+G(0,y)H(1,0,1,z)-\zeta_3G(1,y)\Big)\\
&+2H(1,1,0,1,z)\Big]\\
&+\frac{3}{4}\Big[\frac{1}{2}\Big(H(1,0,1,z)+H(0,0,1,z)+H(0,0,z)G(1-z,y)\Big)\\
&+H(1,0,z)G(1-z,y)+G(0,-z,1-z,y)-G(0,-z,y)H(1,z)\\
&-G(0,1-z,y)H(0,z)-H(0,1,0,z)+\zeta_3H(1,z)+\frac{3}{2}\Big(-H(1,1,0,z)+H(0,z)\Big)\Big]\\
&+\frac{1}{36}\Big[-\frac{99G(0,y)}{4}+\frac{9}{2}\Big(-H(0,0,z)-H(1,0,z)\Big)\Big]\\
&+\frac{\pi^2}{36}\Big[3\Big(-G(1,1,y)+G(0,1,y)+H(0,z)G(1-z,y)\Big)\\
&+\frac{3}{2}\Big(-G(1-z,1,y)-G(1,1-z,y)+G(1,y)H(1,z)-G(0,0,y)\\
&+H(0,z)G(1,y)-G(0,y)H(1,z)-H(0,0,z)\Big)\\
&+\frac{9}{2}\Big(-\frac{G(0,y)}{2}-H(0,z)-H(1,z)\Big)-6H(0,1,z)-6H(1,1,z)\\
&+\frac{3H(1,z)G(1-z,y)}{2}\Big]\\
&+\frac{z}{8(1-y)}\Big[+H(0,z)G(1-z,y)-H(0,1,z)+H(1,0,z)-\frac{17H(0,z)}{4}\Big]\\
&+\frac{(z-1)}{4y}\Big[-\frac{\pi^2G(1-z,y)}{3}+\frac{1}{2}\Big(-H(0,0,z)+H(0,z)G(1-z,y)\\
&-H(0,1,z)\Big)+2\Big(-G(1-z,-z,1-z,y)+H(1,z)G(1-z,-z,y)\\
&+H(0,1,z)G(1-z,y)+G(0,-z,1-z,y)-G(0,-z,y)H(1,z)
\end{aligned}
$$

$$
\begin{aligned}
&- G(0,1-z,y)H(0,z) + H(0,0,z)G(1-z,y)\Big) + \frac{5zH(0,z)}{2y} - \frac{\pi^2}{2}\Big] \\
&\quad + \frac{z^2}{8(1-y)^2}\Big(\frac{4(1-y)}{z} - 1\Big)\Big[- G(1-z,-z,1-z,y) + H(1,1,0,z) \\
&- H(1,0,1,z) - H(1,0,z)G(1-z,y) + H(1,z)G(1-z,-z,y) + H(0,1,0,z) \\
&+ H(0,1,z)G(1-z,y) - H(0,0,1,z) - G(0,-z,1-z,y) + G(0,-z,y)H(1,z) \\
&+ G(0,1-z,y)H(0,z) + \frac{\pi^2}{6}(- G(1-z,y) + H(1,z) + H(0,z))\Big] \\
&\quad + \frac{1}{8y^2}\Big(1 - 2z + z^2\Big)\Big[- G(1-z,-z,1-z,y) + H(1,z)G(1-z,-z,y) \\
&+ H(0,1,z)G(1-z,y) + G(0,-z,1-z,y) - G(0,-z,y)H(1,z) \\
&- G(0,1-z,y)H(0,z) + H(0,0,z)G(1-z,y) + \frac{5z}{y}\Big(G(-z,1-z,y) \\
&- H(1,z)G(-z,y) - H(0,z)G(1-z,y)\Big) + 5\Big(- G(1-z,y) + H(1,z)\Big) \\
&- \frac{\pi^2 G(1-z,y)}{6}\Big] \\
&\quad + \frac{1}{36}\Big(- \frac{153}{4} + \frac{1}{y}(- \frac{9}{2z} + \frac{27}{2} - 9z) + \frac{45}{4z^2} + \frac{45}{2z} + \frac{3\pi^2}{2} - \frac{45y}{2z^2} - \frac{18y}{z} \\
&+ \frac{45y^2}{4z^2}\Big)\Big[+ G(1-z,0,y) - G(1,0,y) + G(0,1-z,y) + H(0,1,z) \\
&- G(0,y)H(1,z) + H(1,z)G(-z,y) - G(-z,1-z,y)\Big] \\
&\quad + \frac{1}{36}\Big(\frac{135}{4} - \frac{1}{y^2}(\frac{9}{4} + \frac{99z}{2} - \frac{207z^2}{4}) + \frac{1}{y}(- 36 + 63z + \frac{9z^2}{2}) \\
&+ \frac{9z(1+z)}{2(1-y)} + \frac{3\pi^2}{2}\Big)\Big[+ G(-z,1-z,y) - H(1,z)G(-z,y) - H(0,z)G(1-z,y)\Big] \\
&\quad + \frac{1}{36}\Big(- \frac{153}{8} + \frac{1}{y}(\frac{45}{4z} + \frac{153}{4} - \frac{99z}{2}) - \frac{45}{2z} - 72\zeta_3 + \frac{27\pi^2}{4} + \frac{45y}{4z}\Big) \times \\
&\Big[G(1-z,y) - H(1,z)\Big] \\
&\quad + \frac{1}{36}\Big(27 - \frac{9z}{2y} + \frac{9\pi^2}{2} + \frac{27z}{2(1-y-z)}\Big)\Big[G(1,0,y) - H(1,0,z) \\
&- G(0,y)H(0,z)\Big] \\
&\quad - \frac{1}{36}\Big(\frac{45(y-1)}{4z} - 36\zeta_3\Big)G(0,y) \\
&\quad - \frac{1}{36}\Big(\frac{1}{y}(\frac{9}{4} - \frac{99z}{2}) - \frac{153z}{8(1-y)} + \frac{153}{4(1-z)} - 36\zeta_3\Big)H(0,z) \\
&\quad - \frac{1}{36}\Big(\frac{3015}{32} + \frac{1}{y}(- \frac{45}{2} + \frac{21\pi^2}{4} + \frac{45z}{2} - 6z\pi^2) + \frac{1}{1-y}(- \frac{27z^2\zeta_3}{2(1-y)} \\
&+ 54z\zeta_3 - \frac{3z\pi^2}{4}) + \frac{621\zeta_4}{4} - \frac{243\zeta_3}{2} + \frac{123\pi^2}{8} + \frac{9z\pi^2}{4(1-y-z)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad +i\pi\Big[-\frac{\pi^2}{8}+\frac{3\zeta_3}{2}+\frac{3}{32}\Big]\,, && \text{(B.15)}\\
D_\gamma^{(2)} &= \frac{1}{36}\Big[\frac{3}{2}\Big(H(0,z)G(1,0,y)-G(0,y)H(1,0,z)\Big)+6\Big(G(1,1,0,y)+H(1,1,0,z)\Big)\\
&\qquad +9\Big(-H(1,0,0,z)+G(1,0,0,y)-H(0,0,z)G(0,y)-G(0,0,y)H(0,z)\Big)\\
&\qquad +29\Big(H(1,0,z)-G(1,0,y)+G(0,y)H(0,z)\Big)+\frac{15}{2}\Big(-H(0,1,0,z)\\
&\quad +G(0,1,0,y)\Big)+\frac{39}{4}\Big(-G(0,y)-H(0,z)\Big)+\frac{155}{6}\Big(H(1,z)-G(1-z,y)\Big)\\
&\qquad +\pi^2\Big(-G(1,y)+\frac{7H(1,z)}{4}-\frac{3G(1-z,y)}{4}\Big)\Big]+\frac{1}{36}\Big(\frac{15z}{4y}+\frac{\pi^2}{4}\Big)\times\\
&\quad \Big[-G(0,y)-H(0,z)\Big]+\frac{1}{4(y+z)^2}\Big[H(1,z)-G(1-z,y)\Big]\\
&\quad +\frac{1}{36y}\Big[\frac{15G(0,y)}{4}+\frac{3H(0,z)}{4}\Big]+\frac{H(0,z)}{12(1-z)}-\frac{1}{36}\Big(-\frac{2393}{72}+\frac{19(1-z)}{y}\\
&\quad +\frac{9}{y+z}-\frac{119\zeta_3}{2}-\frac{79\pi^2}{12}\Big)\\
&\qquad +i\pi\Big[\frac{5}{12}\Big(+H(0,z)*G(0,y)+H(1,0,z)-G(1,0,y)\Big)+\frac{13\pi^2}{144}\\
&\quad +\frac{34}{27}+\frac{5(z-1)}{24y}\Big]\,, && \text{(B.16)}\\
E_\gamma^{(2)} &= \frac{1}{6}\Big[\frac{1}{4}\Big(-H(0,z)G(1,0,y)-G(0,y)H(1,0,z)+H(0,1,0,z)\Big)\\
&\qquad +\frac{1}{2}\Big(-G(-z,1-z,0,y)-G(-z,0,1-z,y)+H(1,0,z)G(-z,y)\\
&\quad +H(1,z)G(-z,0,y)-G(0,-z,1-z,y)+G(0,-z,y)H(1,z)+H(0,0,z)G(0,y)\\
&\quad -H(0,z)G(-z,1-z,y)+H(0,z)G(1-z,0,y)\Big)\\
&\qquad -G(1,1,0,y)-H(1,0,z)G(1-z,y)\\
&\qquad +\frac{3}{2}\Big(G(1-z,0,0,y)-G(1,0,0,y)-G(0,1-z,y)H(0,z)+G(0,1-z,0,y)\\
&\quad -H(0,1,z)G(-z,y)+G(0,0,1-z,y)+H(0,0,z)G(1-z,y)-G(0,0,y)H(1,z)\Big)\\
&\qquad +2\Big(G(-z,-z,1-z,y)-G(1-z,1,0,y)-H(1,z)G(-z,-z,y)\\
&\quad +H(0,1,z)G(0,y)\Big)\\
&\qquad +3\Big(G(1-z,1-z,0,y)+G(1-z,0,1-z,y)-H(1,0,1,z)\\
&\quad -H(1,z)G(1-z,0,y)+G(0,1-z,1-z,y)-G(0,1-z,y)H(1,z)-H(0,1,1,z)\\
&\quad +H(0,1,z)G(1-z,y)+H(0,z)G(1-z,1-z,y)+G(0,y)H(1,1,z)\Big)\\
&\quad -4H(0,0,z)G(0,y)+6\Big(-G(-z,1-z,1-z,y)-G(1-z,-z,1-z,y)\\
&\quad -H(1,1,z)G(-z,y)+H(1,z)G(-z,1-z,y)+H(1,z)G(1-z,-z,y)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{1}{36}\Big[61\Big(G(1-z,1-z,y)+H(1,1,z)-H(1,z)G(1-z,y)\Big) \\
&+21H(0,0,z)G(0,y)-\frac{15G(0,1,0,y)}{2}-\frac{53H(1,0,z)}{4}-15H(0,0,1,z) \\
&+29G(1,0,y)+5G(0,y)H(0,z)+\frac{169}{4}\Big(-H(0,1,z)+H(0,z)G(1-z,y)\Big) \\
&+\frac{41}{2}\Big(+H(0,0,z)+G(0,0,y)\Big)+\frac{63}{4}\Big(-G(1-z,0,y)-G(0,1-z,y) \\
&+G(0,y)H(1,z)\Big)+\pi^2\Big(+G(1,y)-H(1,z)\Big)\Big] \\
&\quad +\frac{1}{4z^2}\Big(1-2y+y^2\Big)\Big[G(1-z,0,y)-G(1,0,y)+G(0,1-z,y)+H(0,1,z) \\
&-G(0,y)H(1,z)-G(-z,1-z,y)+H(1,z)G(-z,y)\Big] \\
&\quad +\frac{z}{2y^3}\Big(1-2z+z^2\Big)\Big[G(-z,1-z,y)-H(1,z)G(-z,y)-H(0,z)G(1-z,y) \\
&-\frac{G(1-z,y)y}{z}+\frac{H(1,z)y}{z}\Big] \\
&\quad +\frac{1}{36}\Big(58+\frac{1}{y^2}(-6-18z+24z^2)+\frac{1}{y}(-3+15z)\Big)\Big[+G(-z,1-z,y) \\
&-H(1,z)G(-z,y)-H(0,z)G(1-z,y)\Big] \\
&\quad +\frac{1}{36}\Big(-\frac{1093}{6}+\frac{9}{y}(\frac{1}{z}+1)-18(\frac{1}{z}+\frac{z}{y})+\frac{1}{y+z}(\frac{39}{y+z}+6)+\frac{9y}{z}\Big)\times \\
&\Big[G(1-z,y)-H(1,z)\Big] \\
&\quad -\frac{1}{36}\Big(\frac{655}{12}-\frac{15(1-z)}{4y}-\frac{9}{z}+\frac{3\pi^2}{4}+\frac{9y}{z}\Big)G(0,y) \\
&\quad -\frac{1}{36}\Big(\frac{547}{12}+\frac{18z(1-z)}{y^2}+\frac{1}{y}(-\frac{27}{4}-\frac{57z}{4})+\frac{3}{1-z}+\frac{3\pi^2}{4}\Big)H(0,z) \\
&\quad -\frac{1}{36}\Big(-\frac{8737}{24}+\frac{37(1-z)}{y}-\frac{39}{y+z}-43\zeta_3+\frac{1975\pi^2}{24}\Big) \\
&\quad +i\pi\Big[\frac{5}{12}\Big(-H(0,z)G(1-z,y)-H(0,1,z)-2H(1,z)G(-z,y) \\
&+H(1,z)G(0,y)-G(1-z,0,y)+2G(-z,1-z,y)-G(0,1-z,y) \\
&+G(1,0,y)\Big)+\frac{155}{144}\Big(-H(0,z)-G(0,y)\Big)+\frac{175}{72}\Big(H(1,z)-G(1-z,y)\Big) \\
&-\frac{\pi^2}{72}+\frac{847}{108}+\frac{5(z-1)}{24y}\Big], \qquad \text{(B.17)}
\end{aligned}
$$

$$
\begin{aligned}
F_\gamma^{(2)} = &\frac{1}{36}\Big[8\Big(-G(1-z,1-z,y)-H(1,1,z)+H(1,z)G(1-z,y)\Big) \\
&\quad +\frac{10}{3}\Big(H(0,z)+G(0,y)\Big)+\frac{40}{3}\Big(G(1-z,y)-H(1,z)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\Big(-G(0,y)H(0,z)-5G(0,0,y)-5H(0,0,z)\Big)\\
&-G(1-z,0,y)+H(1,0,z)-G(0,1-z,y)+H(0,1,z)\\
&-H(0,z)G(1-z,y)+G(0,y)H(1,z)-\frac{100}{9}+\frac{89\pi^2}{12}\Bigg]\\
&+i\frac{5\pi}{72}\Bigg[+H(0,z)-4H(1,z)+4G(1-z,y)+G(0,y)-8\Bigg]\,. \qquad \text{(B.18)}
\end{aligned}
$$

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